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CONTRIBUTIONS TO UNIVERSAL ALGEBRA

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PREFACE

The Bolyai János Mathematical Society hosted a Colloquium on Universal Algebra at the József Attila University in Szeged from August 26 to August 29, 1975. Since 1971, this was the fifth algebra conference held in Szeged and the second on universal algebra.

As one of the 76 participants from 12 countries, including 20 from Hungary, the undersigned feels urged to express our gratitude to all people having contributed to the enterprise. Thanks, indeed, to Professor B. Csákány and the staff members of the Algebra Department: F. Gécseg, A.P. Huhn, L. Klukovits, A. Lenkehegyi, L. Megyesi, G. Pollák, Á. Szendrei, M. Szendrei, and Zs. Turáni. Our thanks include, of course, all persons from the University who kindly helped, we understand, in every respect to bring about the memorable success the meeting turned out to be.

Its scientific program was fairly compact, consisting of 43 talks of various duration. There were, indeed, very impressive and inspiring presentations of well-established people as well as young debutants, demonstrating how very much alife the young field of universal algebra still is. In addition, there was an open-air problem session, held at sunset under the darkening trees of the University's Botanical Gardens, enlightened only by various spontaneous contributions and the good spirits of all. There was some sight-seeing through the old town of Szeged, including some organ music in the striking cathedral, and there was a dinner in the Students' Club accompanied by, and ending in, plenty of Hungarian wine. Szeged, amidst heavily fertile country, on the banks of the majestic and somewhat mythical Theiss river, struck us as a place from the 19-th century where distances, the pace of living, the general setting are still human, enhancing an atmosphere of intellectual creativity, of friendly neighborhood, of inobtrusive enjoyment of life.

The present volume contains 41 papers, 30 of which are written – in some cases expanded – variants of talks delivered at the Colloquium. Five participants sent in papers on topics different from their talks since the latter were to appear elsewhere. Finally, there are six papers included in these Proceedings which were originally planned to be part of the program but could not be presented in person for various reasons.

Most of the papers contain results in the general theory of algebraic systems. The others deal with problems on special kinds of algebras (as groupoids, semigroups, groups, and lattices) which are connected with, suggested by, or suggesting, universal algebraic notions. Each article has been refereed by another participant. The careful and efficient work of many referees is acknowledged with pleasure.

The undersigned, honored by the invitation to act as co-editor of these Proceedings, takes once more the qpportunity to say thanks to our Hungarian hosts and friends for a memorable event of high scientific value, for an informal atmosphere of friendship and understanding.

Jürgen Schmidt

SCIENTIFIC PROGRAM

August 26. Tuesday

Morning session
Chairman: Jürgen V. Schmidt

- I.I. Valutse: Algebras of mappings
- H. Werner: Varieties generated by quasi-primal algebras have decidable theories
- H. Hule: An embedding problem of polynomial algebras
- G. Eigenthaler: On free algebras and algebras of polynomials
- J.C.G. Varlet: Remarks on fully invariant congruences
- E. Nelson – B. Banaschewski: Elementary properties of limit reduced powers with applications to Boolean powers
- L. Lovász: Lifting of structures and homomorphisms of direct products
- D.K. Haley: Equational compactness in rings and applications to topological rings
- H.-J. Bandelt: My last remark on congruence lattices of two-valued algebras

Afternoon session
Chairman: Milan Kolibiar

- J. Ježek – T. Kepka: A survey of our recent and contemporary results in universal algebra
- B. Bosbach: Residuation groupoids
- A.S. Iskander: Covering relations in the lattices of ring and group varieties
- T. Kepka: Triabelian quasigroups

August 27. Wednesday

Morning session
Chairman: E.T. Schmidt

R. Mlitz: Jacobson's density theorem in universal algebra
J. Wiesenbauer: On polynomial completeness in universal algebra
H.K. Kaiser: On completeness problems in universal algebra
A. Iwanik: On semigroups generated by idempotents
D. Schweigert: On automorphical and endomorphical complete algebras
L. Babai: Groups represented as automorphism groups with a small number of orbits
H. Andréka – I. Németi: On related structures, in case of regular arities; closure systems
Z. Páter: Topological algebras of functions
I. Žembery: Proper and improper free algebras
Á. Szendrei: Idempotent reducts of groups and modules

Afternoon session
Chairman: Evelyn Nelson
Problems session

August 28. Thursday

Chairman: Rudolf Wille

J. Schmidt: The lattice-theoretic triple construction and its generalizations

T. Katriňák: On a problem of G. Grätzer

R. Freese: Varieties of modular lattices not generated by their finite dimensional members

E.T. Schmidt: Lattices generated by partial lattices

P. Mederly: Characterization of complete modular p-algebras

J.B. Nation: Results concerning finite sublattices of free lattices

R. Wille: Lattices generated by posets

P. Pudlák: The quadricles – a tool for investigation of lattices (read by P. Goralčik)

E. Graczyńska: On the sums of double systems of lattices and DS-congruences in lattices

August 29. Friday

Chairman: Ralph S. Freese

M. Kolibiar: Primitive subsets of algebras

R. Quackenbush: Varieties with small fine spectra

A. Day: Varieties of algebras that are congruence modular

J. Timm: On regular algebras

E. Beutler: The c-ideal lattice and subalgebra lattice are independent

M. Steinby: On algebras as tree automata

S. Fajtlowicz: Duality for algebras

A. Petrescu: Certain questions of the theory of homotopy of universal algebras

M. Sekanina: Concrete categories with non-injective monomorphisms

R. John: Notions for validity of equations in partial algebras

M. Münzová-Demlová: Epimorphisms in universal algebras

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COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
17. CONTRIBUTIONS TO UNIVERSAL ALGEBRA, SZEGED (HUNGARY), 1975.

ON THE CONGRUENCE LATTICE OF PSEUDO-SIMPLE ALGEBRAS

H. ANDRÉKA – I. NÉMETI

From notions, defined by the existence of proper congruences – e.g. simplicity, direct indecomposability, weak subdirect indecomposability – we obtain the corresponding pseudo notions – e.g. pseudo-simplicity, direct pseudo-indecomposability, weak subdirect pseudo-indecomposability – by changing from "the existence of proper congruences" to "the existence of nonisomorphic factoralgebras" (see e.g. Monk [3], Szélpál [4], Henkin – Monk – Tarski [1] pages 77, 93, 104, Remarks 0.3.57 and 0.3.59.).

Szélpál [4] gave a complete characterization of pseudo-simple Abelian groups. Monk [3] proved that the congruence lattice of any pseudo-simple algebra is isomorphic to an ordinal (see also Henkin – Monk – Tarski [1], p. 77). Here we prove that the congruence lattice of any pseudo-simple algebra is isomorphic to an indecomposable ordinal's successor, and vice versa: to any indecomposable ordinal α, there is a finitary algebra with congruence lattice isomorphic to $\alpha + 1$. By a counterexample we also show that more cannot be said – in a certain sense at least.

Definition. The powers of the ordinal ω are said to be the indecomposable ordinals, that is, an ordinal α is called indecomposable if $\alpha = \omega^{\beta}$ for some ordinal β, where ω is the set of all natural numbers (cf. Kuratowski [2], p. 259, Th. 7).

Definition. An algebra $\mathfrak{A}$ is called pseudo-simple if for every proper congruence $R \neq A \times A$ on $\mathfrak{A}$ it holds that

$$\mathfrak{A}/R \cong \mathfrak{A}.$$

Examples of pseudo-simple and related algebras:

1. Let $\mathfrak{A} = \langle \omega, f \rangle$ where $f(n+1) = n$ and $f(0) = 0$. (See Fig. 1.) Clearly, $\mathfrak{A}$ is pseudo-simple but not simple. For more examples see e.g. Henkin – Monk – Tarski [1].

2. A denumerable atomless Boolean algebra is finitely directly pseudo-indecomposable; a Boolean group of cardinality continuum is finitely directly pseudo-indecomposable, but neither finitely directly indecomposable, nor infinitely directly pseudo-indecomposable; every infinite algebra in which the number of operations is zero (or the operations are trivial) is weakly subdirectly pseudo-indecomposable, etc.

Fig. 1

Let $C(\mathfrak{A}) = \langle C(\mathfrak{A}), \subseteq \rangle$ denote the congruence lattice of the algebra $\mathfrak{A}$.

Theorem 1. *The lattice L is isomorphic to the successor of an in-*

decomposable ordinal (there is an indecomposable ordinal α *for which* $L \cong \langle \alpha + 1, \leqslant \rangle$) *iff there exists a pseudo-simple algebra* $\mathfrak{A}$ *such that* $C(\mathfrak{A}) \cong L$.

Proof.

1. Let $\mathfrak{A}$ be pseudo-simple. Now $C(\mathfrak{A})$ is clearly isomorphic to all of its principal dual ideals. By Monk [3], $C(\mathfrak{A})$ is a well-ordering and thus isomorphic to an ordinal, but any ordinal which is isomorphic to all of its principal dual ideals (or right segments), is indecomposable (cf. [2], p. 259, Th. 7).

2. Let α be an indecomposable ordinal, $\mathfrak{A} = \langle \alpha, f \rangle$, where for any $\rho, \mu, \nu \in \alpha$,

$$f(\rho, \mu, \nu) = \begin{cases} 0 & \text{if} \quad \rho > \mu, \nu, \\ \nu & \text{otherwise.} \end{cases}$$

The identity relation on α is denoted by Id. Now we have

(1) $\qquad C(\mathfrak{A}) \subseteq \{\gamma \times \gamma \cup \mathrm{Id}\colon \gamma \leqslant \alpha\}$,

since for any congruence R on $\mathfrak{A}$, if $a R b$, $b \neq a$ and $c < \max(a, b)$, then $c R 0$ by the definition of f.

For any $\gamma < \alpha$ let j_γ be an isomorphism from the dual ideal generated by γ onto α, that is from $\langle \alpha \setminus \gamma, \leqslant \rangle$ onto $\langle \alpha, \leqslant \rangle$. Since α is indecomposable, there exists such an isomorphism (cf. [2], p. 259).

For every $0 < \gamma \leqslant \alpha$ we define

$$i_\gamma(\eta) = \begin{cases} j_\gamma(\eta) + 1 & \text{if} \quad \eta \geqslant \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

(See Fig. 2)

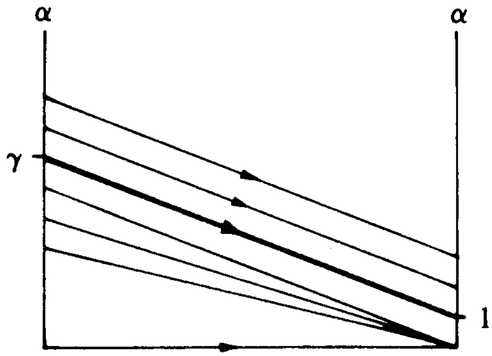

Fig. 2

To see that i_γ is an endomorphism, observe that, using the abbreviation $i_\gamma(x) = x'$, the identity $f(\rho', \mu', \nu') = (f(\rho, \mu, \nu))'$ is valid in $\mathfrak{A}$, since

(a) if $\rho' > \mu', \nu'$ then $\rho > \mu, \nu$ and thus both sides are zero,

(b) if $\rho' \not> \mu', \nu'$ then either $\rho \not> \mu, \nu$ and thus both sides are ν', or $\rho > \mu, \nu$ and $\nu' = 0$ thus both sides are 0.

Observe that for any $\gamma \neq \alpha$ the function i_γ is onto, since j_γ is onto. Now (1) implies that every congruence is the kernel of some i_γ, that is,

$$C(\mathfrak{A}) = \{\gamma \times \gamma \cup \mathrm{Id}\colon\ \gamma \leqslant \alpha\}.$$

But if $\gamma \neq 0, \alpha$ then i_γ induces an isomorphism between $\mathfrak{A}/\mathrm{Ker}\, i_\gamma$ and $\mathfrak{A}$, proving that $\mathfrak{A}$ is pseudo-simple. Further we have

$$C(\mathfrak{A}) = \langle\{\gamma \times \gamma \cup \mathrm{Id}\colon\ \gamma \leqslant \alpha\}, \subseteq\rangle \cong \langle\{\gamma\colon\ \gamma \leqslant \alpha\}, \subseteq\rangle = \langle\alpha + 1, \leqslant\rangle,$$

which proves the theorem.

The above theorem states only that to any indecomposable ordinal α there exists a pseudo-simple algebra $\mathfrak{A}$ such that $C(\mathfrak{A}) \cong \langle\alpha + 1, \leqslant\rangle$. Naturally, the question arises what can be deduced from the information that $C(\mathfrak{A})$ is (the successor of) an indecomposable cardinal. The following counterexample shows that it does not imply pseudo-simplicity.

Theorem 2. *To every ordinal* α *there is an algebra* $\mathfrak{B}$ *such that* $C(\mathfrak{B}) \cong \langle \alpha + 1, \leqslant \rangle$ *and* $\mathfrak{B}$ *is not pseudo-simple.*

Proof. Let $\mathfrak{B} = \langle \alpha, f \rangle$, where for any $\rho, \mu, \nu \in \alpha$ we define

$$f(\rho, \mu, \nu) = \begin{cases} 1 & \text{if} \quad \rho > \mu, \nu, \\ \nu & \text{otherwise.} \end{cases}$$

Clearly, $\mathfrak{B}$ is not pseudo-simple, since the algebra $\mathfrak{A}$ defined in the previous proof is a homomorphic image of $\mathfrak{B}$, but is by no means isomorphic to it. (Observe that $f(0, 1, 1) = 1 = f(1, 0, 0)$ holds in $\mathfrak{B}$ but in $\mathfrak{B}/2 \times 2 \cup \mathrm{Id} \cong \mathfrak{A}$ there are no distinct elements a, b such that $f(a, b, b) = f(b, a, a)$.)

To see that $C(\mathfrak{B}) \cong \langle \alpha + 1, \leqslant \rangle$, we should repeat the corresponding parts of the previous proof. Here i_γ is not an endomorphism but only a homomorphism, since the operation of the image-algebra differs from the operation of $\mathfrak{B}$; and also if α is not indecomposable then i_γ is not onto either. But, since now we do not want to prove pseudo-simplicity, it is enough to show that $\mathrm{Ker}\, i_\gamma$ is a congruence on $\mathfrak{B}$.

Now we show that $C(\mathfrak{B}) = \{\gamma \times \gamma \cup \mathrm{Id}\colon \gamma \leqslant \alpha\}$.

(i) $C(\mathfrak{B}) \subseteq \{\gamma \times \gamma \cup \mathrm{Id}\colon \gamma \leqslant \alpha\}$, since for any congruence R on $\mathfrak{B}$, if $a R b$, $b \neq a$, and $c < \max(a, b)$, then $c R 1$ by the definition of f. Substituting 0 for c, we also obtain $0 R 1$.

(ii) It remains to show that $\gamma \times \gamma \cup \mathrm{Id}$ is a congruence on $\mathfrak{B}$ if $\gamma \leqslant \alpha$. Denote the equivalence $\gamma \times \gamma \cup \mathrm{Id}$ by $\equiv$ and suppose $\equiv \neq \mathrm{Id}$. Let $\rho \equiv \rho'$, $\mu \equiv \mu'$, $\nu = \nu'$, and put $\varphi = f(\rho, \mu, \nu)$, $\varphi' = f(\rho', \mu', \nu')$. We have to show that $\varphi \equiv \varphi'$. The hypothesis $\varphi \neq \varphi'$ implies $\varphi' = 1$, $\varphi = \nu$ (or vice versa) and $\nu \not\equiv 1$. $\nu \not\equiv 1$ implies $\nu \geqslant \gamma$ and $\nu = \nu'$. Then $\varphi' = 1$ implies $\rho = \rho' > \nu' = \nu \geqslant \gamma$. But now $\varphi = \nu$ implies $\mu \geqslant \rho > \gamma$ and thus $\mu = \mu'$, which contradicts the hypothesis $\varphi = \varphi'$.