Lattices, Semigroups, and Universal Algebra

Edited by
Jorge Almeida,
Gabriela Bordalo, and
Philip Dwinger

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Proceedings of an international conference on Lattices, Semigroups, and Universal Algebra, held June 20-24, 1988, in Lisbon, Portugal

PREFACE

This volume contains papers which, for the most part, are based on talks given at an international conference on Lattices, Semigroups, and Universal Algebra that was held in Lisbon, Portugal during the week of June 20-24, 1988. The conference was dedicated to the memory of Professor António Almeida Costa, a Portuguese mathematician who greatly contributed to the development of algebra in Portugal, on the 10^{th} anniversary of his death. The themes of the conference reflect some of his research interests and those of his students.

The purpose of the conference was to gather leading experts in Lattices, Semigroups, and Universal Algebra and to promote a discussion of recent developments and trends in these areas. All three fields have grown rapidly during the last few decades with varying degrees of interaction. Lattice theory and Universal Algebra have historically evolved alongside with a large overlap between the groups of researchers in the two fields. More recently, techniques and ideas of these theories have been used extensively in the theory of semigroups. Conversely, some developments in that area may inspire further developments in Universal Algebra. On the other hand, techniques of semigroup theory have naturally been employed in the study of semilattices. Several papers in this volume elaborate on these interactions.

The conference started with a public session during which J. Morgado addressed the historical significance of Almeida Costa's work on behalf of algebra in Portugal and H. J. Weinert discussed the recent developments connected with his research. The scientific program also included invited survey talks, contributed papers — some which were also presented by invitation — and a problem session. The survey talks were given by J. Berman, M. Erné, R. Freese, G. Grätzer, P. Jones, G. Lallement, R. McKenzie, W. D. Munn, J. E. Pin, N. R. Reilly, J. Rhodes, J. Varlet, and H. Werner. The general impression among the participants was that the conference met their expectations both scientifically and socially.

The success of this conference depended on many people. Among them, the members of the scientific committee, J. Berman, J. A. Dias da Silva, Ph. Dwinger, J. Furtado Coelho, M. L. Galvão, G. Grätzer, J. Howie, G. Lallement, R. McFadden, J. Morgado, M. B. Ramalho, J. Varlet, and the members of the Universal Algebra and lattice theory group of the Centro de Álgebra, in particular M. Sequeira, played a role in the preparation of the conference and of these Proceedings which is hereby acknowledged.

The conference took place at the University of Lisbon and was financially supported by the following organizations: "Instituto Nacional de Investigação Científica", "Junta Nacional de Investigação Científica e Tecnológica", "Fundação Calouste Gulbenkian", "Reitoria" of the University of Lisbon, "Conselho Directivo" and Department of Mathematics of the Faculty of Sciences of the University of Lisbon, New University of Lisbon, and "Caixa Geral de Depósitos". The Lisbon City Council ("Câmara Municipal") provided a reception for participants at the Town Hall. Without the contributions of these institutions, the conference could not have taken place.

Oporto, Lisbon, Chicago

July, 1989

Jorge Almeida

Gabriela Bordalo

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CONTENTS

THE JOIN OF THE PSEUDOVARIETY J WITH PERMUTATIVE PSEUDOVARIETIES

Assis de Azevedo

Abstract

Using a theorem of Reiterman, which characterizes pseudovarieties as classes of finite semigroups satisfying a set of pseudoidentities, and a characterization of the implict operations on DS, we calculate some joins of the form $\mathbf{J} \vee \mathbf{V}$, where $\mathbf{V}$ is a permutative pseudovariety. As a consequence we obtain that, for these $\mathbf{V}$, $\mathbf{J} \vee \mathbf{V}$ is decidable if and only if $\mathbf{V} \cap \mathbf{CS}$ is decidable.

1. Introduction

A class of finite semigroups closed under the formation of homomorphic images, subsemigroups and finitary direct produts is said to be a *pseudovariety* of semigroups.

An "implicit operation" on a pseudovariety $\mathbf{V}$ is any new operation whose introduction does not eliminate homomorphisms between elements of $\mathbf{V}$.

Reiterman [8] proved that pseudovarieties are defined by *pseudoidentities*, i.e., by formal equalities of implicit operations.

In this paper, we calculate the join $\mathbf{J} \vee \mathbf{V}$, for a pseudovariety $\mathbf{V}$ such that $\mathbf{W} \subseteq \mathbf{V} \subseteq \mathbf{Com} \vee \mathbf{W}$, where $\mathbf{W}$ is equal to $\mathbf{I}$, $\mathbf{N}$, $\mathbf{K}$, $\mathbf{D}$ or $\mathbf{LI}$. Moreover, we prove that the decidability of $\mathbf{J} \vee \mathbf{V}$ is equivalent to the one of $\mathbf{V} \cap \mathbf{CS}$ (see section 2 for notation).

The calculation of joins of pseudovarieties is not, in general, very easy. Actually the result can be very surprising and in fact the join of very simple decidable pseudovarieties can have an undecidable membership problem [1].

Our proof of the main theorem is based on Reiterman's results and on a characterization of implicit operations on DS, the class of finite semigroups such that the regular $\mathcal{D}$-classes are subsemigroups. The idea is to prove that any pseudoidentity satisfied by **J** and **V**, is also satisfied by a certain subpseudovariety of **DS** containing **J** and **V**, my guess for **J**$\vee$**V**. This kind of approach was used to calculate, for example **G**$\vee$**Com** [4] and **R**$\vee$**L** [6] (see again section 2 for notation).

2. Preliminaries

Let **V** be a pseudovariety.

Definition 2.1 For $n \in \mathbb{N}$, an *n-ary implicit operation on* **V** is given by $\pi = (\pi_S)_{S \in V}$ where

i) for each $S \in V$, $\pi_S : S^n \to S$ is a function;

ii) for each homomorphism $\varphi: S \to T$ with $S,T \in V$, the following diagram commutes:

$$\begin{array}{ccc} S^n & \xrightarrow{\pi_S} & S \\ \downarrow \varphi^n & & \downarrow \varphi \\ T^n & \xrightarrow{\pi_T} & T. \end{array}$$

The set of all n-ary implicit operations on **V** is denoted by $\overline{\Omega}_n V$. $\overline{\Omega}_n V$ has a natural structure of semigroup, defining, for $\pi,\rho \in \overline{\Omega}_n V$ and $S \in V$, $(\pi\rho)_S = \pi_S \rho_S$.

The subsemigroup of $\overline{\Omega}_n V$ generated by $\{x_1,\ldots, x_n\}$ is denoted by $\Omega_n V$ (where$(x_i)_S\ (a_1,\ldots,a_n) = a_i$ for $i \in \{1,\ldots,n\}$, $S \in V$ and $a_1,\ldots, a_n \in S$).

Another example of implicit operation is given by the unary operation $x \mapsto x^\omega$ such that, for a finite semigroup S and $a \in S$, a^ω is the idempotent of the semigroup generated by a.

A formal equality $\pi = \rho$ with π and ρ in $\overline{\Omega}_n V$ $(n \in \mathbb{N})$ is called a *pseudoidentity*. A semigroup $S \in V$ is said to *satisfy* the pseudoidentity $\pi = \rho$, and we write $S \models \pi = \rho$ if $\pi_S = \rho_S$. We will denote $\{x_1,\ldots,x_n\}$ by A.

Reiterman [9] defined a distance on $\overline{\Omega}_n V$ such that $\overline{\Omega}_n V$ is a compact topological semigroup on which $\Omega_n V$ is dense.

The convergence of a sequence $(\pi_m)_{m \in \mathbb{N}}$ in $\overline{\Omega}_n V$ to π is equivalent to the following condition,

$$(\forall k \in \mathbb{N})(\forall S \in V)(\exists p \in \mathbb{N})[m \geq p, |S| \leq k \Rightarrow S \models \pi_m = \pi].$$

Theorem 2.2 (Reiterman [9]) *Let* $\mathcal{C}$ *be any subclass of* **V**. *Then* $\mathcal{C}$ *is a pseudovariety if and only if there is a set* $\sum$ *of pseudovarieties for* **V** *such that*

$$\mathcal{C} = [\![\textstyle\sum]\!] \stackrel{\text{def}}{=} \{S \in V : S \models \pi = \rho,\ \forall(\pi=\rho) \in \textstyle\sum\}.$$

The following list of pseudovarieties will appear in the sequel. **U** denotes a class of finite semigroups, and all semigroups being considered are finite.

I $= \{$trivial semigroups$\} = [\![x = y]\!]$,
Com $= \{$commutative semigroups$\} = [\![xy = yx]\!]$,

Se $= \{\text{semilattices}\} = [\![x^2 = x,\ xy = yx]\!]$,
G $= \{\text{groups}\} = [\![\ x^{\omega}y = yx^{\omega} = y]\!]$,
J $= \{\mathcal{J}\text{-trivial semigroups}\} = [\![x^{\omega+1} = x^{\omega},\ (xy)^{\omega} = (yx)^{\omega}]\!]$,
K $= \{\text{semigroups such that idempotents are left zeros}\} = [\![x^{\omega}y = x^{\omega}]\!]$,
D $= \{\text{semigroups such that idempotents are right zeros}\} = [\![yx^{\omega} = x^{\omega}]\!]$,
LI $= \{\text{locally trivial semigroups}\} = [\![x^{\omega}yx^{\omega} = x^{\omega}]\!]$,
CS $= \{\text{completely simple semigroups}\} = [\![(x^{\omega}yx^{\omega})^{\omega} = x^{\omega},\ x^{\omega+1} = x]\!]$,
DU $= \{\text{semigroups such that regular } \mathcal{D}\text{-classes are semigroups of U}\}$,
DS $= \{\text{semigroups such that regular } \mathcal{D}\text{-classes are semigroups}\}$
$= [\![((xy)^{\omega}(yx)^{\omega}(xy)^{\omega})^{\omega} = (xy)^{\omega}]\!]$,
DO $= \{\text{semigroups such that regular } \mathcal{D}\text{-classes are orthodox semigroups}\}$
$= [\![(xy)^{\omega}(yx)^{\omega}(xy)^{\omega} = (xy)^{\omega}]\!]$.

Definition 2.3 For $\pi \in \overline{\Omega}_n V$, let $c(\pi)$, the content of π, denote the set of all $x_i \in A$ such that, for some $S \in V$, π_S depends on the i^{th} component.

Proposition 2.4 ([8]) If $Se \subseteq V$ then, considering $\mathcal{P}(A)$ as a discrete semilattice under union, the mapping

$$c : \overline{\Omega}_n V \to \mathcal{P}(A)$$

is the only continuous homomorphism for which $c(x_i) = \{x_i\}$.

The importance of this function is justified by the following.

Lemma 2.5 ([7]) For $\pi,\rho \in \overline{\Omega}_n DS$, π^{ω} is $\mathcal{J}$-equivalent to p^{ω} if and only if $c(\pi) = c(\rho)$. In particular $\overline{\Omega}_n DS$ has $2^n - 1$ regular $\mathcal{J}$-classes.

The results presented here depend on two theorems, which give a decomposition of implicit operations on J and on DS.

Theorem 2.6 (Almeida [5]) *Let* $\pi,\rho \in \overline{\Omega}_n J$.

i) J *satisfies* $\pi^{\omega} = \rho^{\omega}$ *if and only if* $c(\pi) = c(\rho)$.

ii) *We can write* π *in the form* $u_0 v_1^{\omega} u_1 \cdots v_k^{\omega} u_k$, *with* $u_i, v_j \in A^*$, *such that the first letter of* $u_i \notin c(v_i)$ *and the last letter of* $u_i \notin c(v_{i+1})$. *If* $u_i = 1$ *then the sets* $c(v_i)$ *and* $c(v_{i+1})$ *are incomparable.*

iii) *If* $\pi = u_0 v_1^{\omega} u_1 \cdots v_k^{\omega} u_k$ *and* $\rho = w_0 t_1^{\omega} w_1 \cdots t_s^{\omega} w_s$ *as above then* J *satisfies* $\pi = \rho$ *if and only if* $k = s$, $u_i \equiv w_i$ *and* $c(v_i) = c(t_i)$.

Theorem 2.7 ([7]) *Let* $\pi \in \overline{\Omega}_n DS$. *Then we can write* π *in the form* $u_0 \pi_1^{\omega+1} u_1 \cdots \pi_k^{\omega+1} u_k$, *with* $u_i \in A^*$, *such that the first letter of* $u_i \notin c(\pi_i)$ *and the last letter of* $u_i \notin c(\pi_{i+1})$. *If* $u_i = 1$ *then the sets* $c(\pi_i)$ *and* $c(\pi_{i+1})$ *are incomparable.*

Moreover, if $\pi,\rho \in \overline{\Omega}_n W$ (*with* $W \subseteq DO$) *are regular, then*

$$W \models \pi = \rho \Leftrightarrow W \cap G \models \pi = \rho.$$

3. Results

In [2], Almeida defined the pseudovarieties $\mathbf{Perm}_{k,m,l}$ with $k,l \geq 0$ and $m \geq 2$ as the class of finite semigroups satisfying the identities of the form

$$x_1 \dots x_k y_1 \dots y_m z_1 \dots z_l = x_1 \dots x_k y_{\sigma(1)} \cdots y_{\sigma(m)} z_1 \cdots z_l$$

where σ is a permutation of $\{1, \dots, m\}$. For example, $\mathbf{Perm}_{(0,2,0)}$ is the class of all comutative semigroups. Writing ∞ in a component of the triple index, he means that union is taken over all possible finite values of that component.

Let **Perm** be the class of finite permutative semigroups, i.e., the class of finite semigroups satisfying an identity of the form $x_1 \dots x_n = x_{\sigma(1)} \cdots x_{\sigma(n)}$ for a nontrivial permutation σ.

In [3], Almeida proved that $\mathbf{Perm} = \mathbf{Perm}_{(\infty,\infty,\infty)} = \mathbf{Com} \vee \mathbf{LI}$, $\mathbf{Perm}_{(\infty,2,0)} = \mathbf{Com} \vee \mathbf{K}$ and $\mathbf{Perm}_{(0,2,\infty)} = \mathbf{Com} \vee \mathbf{D}$.

Theorem 3.1. (Main Theorem) i) *If* **V** *is a pseudovariety such that* $\mathbf{W} \subseteq \mathbf{V} \subseteq \mathbf{Com} \vee \mathbf{W}$, *where* **W** *is equal to* **I**, **K**, **D** *or* **LI** *then* $\mathbf{J} \vee \mathbf{V}$ *is equal to*

$$\mathbf{DV} \cap [\![x^\omega y x^{\omega+1} = x^{\omega+1} y x^\omega,\ \gamma_1 = \delta_1,\ (\alpha\beta)^{\omega+1} = \beta(\alpha\beta)^\omega,\ (\epsilon\delta)^{\omega+1} = (\epsilon\delta)^\omega \epsilon]\!]$$

where

$$\alpha = (zt^\omega x)^\omega,\ \epsilon = (xt^\omega z)^\omega,$$
$$\beta = (zt^\omega y)^\omega,\ \delta = (yt^\omega z)^\omega,$$

$$\gamma_1 = \begin{cases} x^\omega a(st)^\omega b y^\omega & \text{if } \mathbf{LI} \subseteq \mathbf{V} \subseteq \mathbf{Perm} \\ x^\omega a(st)^\omega & \text{if } \mathbf{K} \subseteq \mathbf{V} \subseteq \mathbf{Perm}_{(\infty,2,0)} \\ (st)^\omega b y^\omega & \text{if } \mathbf{D} \subseteq \mathbf{V} \subseteq \mathbf{Perm}_{(0,2,\infty)} \\ 1 & \text{if } \mathbf{V} \subseteq \mathbf{Com} \end{cases}$$

and

$$\delta_1 = \begin{cases} x^\omega a(ts)^\omega b y^\omega & \text{if } \mathbf{LI} \subseteq \mathbf{V} \subseteq \mathbf{Perm} \\ x^\omega a(ts)^\omega & \text{if } \mathbf{K} \subseteq \mathbf{V} \subseteq \mathbf{Perm}_{(\infty,2,0)} \\ (ts)^\omega b y^\omega & \text{if } \mathbf{D} \subseteq \mathbf{V} \subseteq \mathbf{Perm}_{(0,2,\infty)} \\ 1 & \text{if } \mathbf{V} \subseteq \mathbf{Com}. \end{cases}$$

Moreover, if $\mathbf{V} \subseteq \mathbf{Com}$ *the pseudoidentities* $(\alpha\beta)^{\omega+1} = \beta(\alpha\beta)^\omega$ *and* $(\epsilon\delta)^{\omega+1} = (\epsilon\delta)^\omega \epsilon$ *may be omitted. The same happens if* $\mathbf{V} \subseteq \mathbf{Perm}_{(\infty,2,0)}$ *or* $\mathbf{V} \subseteq \mathbf{Perm}_{(0,2,\infty)}$ *relatively to the pseudoidentities* $(\epsilon\delta)^{\omega+1} = (\epsilon\delta)^\omega \epsilon$ *or* $(\alpha\beta)^{\omega+1} = \beta(\alpha\beta)^\omega$, *respectively.*

ii) *If* $\mathbf{V}$ *is a pseudovariety such that* $\mathbf{N} \subseteq \mathbf{V} \subseteq \mathbf{Com} \vee \mathbf{N}$ *then* $\mathbf{J} \vee \mathbf{V}$ *is equal to* $\mathbf{J} \vee (\mathbf{V} \cap \mathbf{Com})$.

iii) *For a pseudovariety as in* i) *or* ii), $\mathbf{J} \vee \mathbf{V}$ *is decidable if and only if* $\mathbf{V} \cap \mathbf{CS}$ *is decidable.*

For the proof of Theorem 3.1 we need some preliminary results.

Lemma 3.2. If $\mathbf{W}$ is a subpseudovariety of $\mathbf{CS}$ then, for any pseudovariety $\mathbf{U}$, $(\mathbf{U} \vee \mathbf{J}) \cap \mathbf{W}$ is equal to $\mathbf{U} \cap \mathbf{W}$.

Proof. Let $S \in (\mathbf{U} \vee \mathbf{J}) \cap \mathbf{W}$. Then there exists a diagram

$$\begin{array}{ccc} T & \overset{i}{\hookrightarrow} & U \times J \\ \downarrow \varphi & & \\ S & & \end{array}$$

such that $U \in \mathbf{U}$, $J \in \mathbf{J}$ and φ is a surmorphism.

If I is the minimum ideal of T then, as S is completely simple, $\varphi(I) = S$. Then we only need to prove that $I \in \mathbf{U}$.

Let $\pi_2 : V \times J \rightarrow J$ be the projection on the second component. As I is completely simple and J is $\mathcal{J}$-trivial $\pi_2 i(I)$ is a trivial semigroup. Then $I \in \mathbf{U}$ because it is isomorphic to a subsemigroup of U. □

Lemma 3.3 Let $\mathbf{V}$ be any pseudovariety and $\alpha, \beta \in \overline{\Omega}_1\mathbf{V}$ be such that $\alpha = \alpha^{\omega+1}$ and $\beta = \beta^{\omega+1}$. Then

$$\mathbf{V} \models \alpha = \beta \Leftrightarrow \mathbf{V} \cap \mathbf{G} \models \alpha = \beta.$$

Proof. Just note that for $S \in \mathbf{V}$ and $a \in S$, $\alpha_S(a) = \alpha_{\mathcal{H}_{a^\omega}}(a^{\omega+1})$ and $\beta_S(a) = \beta_{\mathcal{H}_{a^\omega}}(a^{\omega+1})$ where $\mathcal{H}_a$ denotes the $\mathcal{H}$-class of a. □

Lemma 3.4. Let $S \in \mathbf{DO}$, e, f, a, $b \in S$ with e and f idempotents and $e \leq_J f, a, b$. Then eabe = eafbe.

Proof. As ea and be are group elements (since they lie in the $\mathcal{J}$-class of e) we have,

$$\begin{aligned} eabe &= ea.(ea)^\omega(be)^\omega.be = ea.e.be \\ &= ea.(ea)^\omega f(be)^\omega.be = (ea)^{\omega+1}f(be)^{\omega+1} = eafbe. \ \square \end{aligned}$$

Lemma 3.5 Let π, ρ, $\alpha_1, \ldots, \alpha_k \in \overline{\Omega}_n\mathbf{DO}$ and σ be a permutation of $\{1, \ldots, k\}$. Then

i) $\mathbf{Perm} \models \pi^\omega \alpha_1 \ldots \alpha_k \pi^\omega = \pi^\omega \alpha_{\sigma(1)} \cdots \alpha_{\sigma(k)} \pi^\omega$,

ii) $\mathbf{DPerm} \models \pi^\omega \alpha_1 \ldots \alpha_k \pi^\omega = \pi^\omega \alpha_{\sigma(1)} \cdots \alpha_{\sigma(k)} \pi^\omega$ if $c(\alpha_i) \subseteq c(\pi)$ $(1 \leq i \leq k)$,

iii) $\mathbf{Perm}, \mathbf{DPerm} \subseteq \mathbf{DO}$.

Proof. For i) it suffices to use the following characterization of permutative semigroups: Perm $= [\![x^{\omega}yzt^{\omega} = x^{\omega}zyt^{\omega}]\!]$.

For ii), by the preceding lemma

$$\begin{aligned}\mathrm{DPerm} \models \pi^{\omega}\alpha_1 \ldots \alpha_k \pi^{\omega} &= \pi^{\omega}(\alpha_1\pi^{\omega})\ldots(\alpha_k\pi^{\omega})\pi^{\omega} \\ &= \pi^{\omega}(\alpha_{\sigma(1)}\pi^{\omega})\ldots(\alpha_{\sigma(k)}\pi^{\omega})\pi^{\omega},\end{aligned}$$

since, for $A \in$ DPerm and $a_1,\ldots,a_n \in A$, we have that

$$\pi^{\omega}_A(a_1,\ldots,a_n),\ (\alpha_1\pi^{\omega})_A(a_1,\ldots,a_n),\ldots,(\alpha_k\pi^{\omega})_A(a_1,\ldots,a_n)$$

are elements of $I_{\pi^{\omega}_A(a_1,\ldots,a_n)}$, which is a permutative semigroup. Applying again the preceding lemma, we obtain

$$\mathrm{DPerm} \models \pi^{\omega}\alpha_1 \ldots \alpha_k \pi^{\omega} = \pi^{\omega}\alpha_{\sigma(1)}\pi^{\omega} \ldots \alpha_{\sigma(k)}\pi^{\omega}.$$

For iii), let S be a semigroup from either Perm or DPerm. If $n \in \mathbb{N}$ is such that $a^{\omega} = a^n$ for all $a \in S$, then, for $a,b \in S$,

$$\begin{aligned}(ab)^{\omega}(ba)^{\omega}(ab)^{\omega} &= (ab)^{\omega}(ba)^{n}(ab)^{\omega} \\ &= (ab)^{\omega}(ab)^{n}(ab)^{\omega} \quad \text{using i) or ii)} \\ &= (ab)^{\omega}(ab)^{\omega}(ab)^{\omega} \\ &= (ab)^{\omega}.\end{aligned}$$

So $S \in$ DO. □

Corollary 3.6 If V is a subpseudovariety of DPerm, $\pi = \pi^{\omega+1} \in \overline{\Omega}_m V$ and $c(\pi) = \{x_1,\ldots, x_k\}$, then there exist $\alpha_1,\ldots, \alpha_k \in \overline{\Omega}_1 V$ such that

$$V \models \pi = \pi^{\omega}\alpha_1(x_1)\ldots\alpha_k(x_k)\pi^{\omega} \quad \text{with } \alpha_i = \alpha_i^{\omega+1}.$$

If $\pi = \pi^{\omega}\beta_1(x_1)\ldots\beta_k(x_k)\pi^{\omega}$ with $\beta_i = \beta_i^{\omega+1}$ is another factorization of π of the same type, then, for all i, $V \models \alpha_i = \beta_i$.

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\Omega_m V$ with limit π. We may suppose that $c(u_n) = c(\pi)$ for all n and, as $\overline{\Omega}_m V$ is compact, that the sequence $(u_n^{(i)})_{n\in\mathbb{N}}$ (for $i \in \{1,\ldots,k\}$), obtained from $(u_n)_{n\in\mathbb{N}}$ by substituting by 1 the variables x_j for $j \neq i$, has a limit β_i. Let $\alpha_i = \beta_i^{\omega+1}$.

Then, in V, we have, using the preceding lemmas and the fact that the semigroups of implicit operations are topological semigroups,

$$\begin{aligned}\pi = \pi^{\omega}\pi\pi^{\omega} &= \lim_n \pi^{\omega}u_n\pi^{\omega} \\ &= \lim_n \pi^{\omega}u_n^{(1)} \ldots u_n^{(k)}\pi^{\omega} \\ &= \pi^{\omega}\lim_n u_n^{(1)} \ldots \lim_n u_n^{(k)}\pi^{\omega}\end{aligned}$$

$$= \pi^\omega \beta_1 \dots \beta_k \pi^\omega$$
$$= \pi^\omega \alpha_1 \dots \alpha_k \pi^\omega.$$

For the second part use Lemma 3.3. □

Definition 3.7 Let $\pi \in \bar{\Omega}_n \mathrm{DPerm}$. Let $\pi = u_0 \pi_1 u_1 \dots \pi_k u_k$ be the factorization given by Theorem 2.7. Write $\pi_i = \pi_i^\omega \alpha_{1,i} \dots \alpha_{k_i,i} \pi_i^\omega$ as in the previous corollary.

Define $\hat{\pi}_i = \pi_i^\omega \tilde{\pi}_i \pi_i^\omega$, where $\tilde{\pi}_i$ is any product of all $\alpha_{s,i}$ such that $c(\alpha_{s,i}) \subseteq c(\pi_i) \setminus \cup_{j<i} c(\pi_j)$ and $\hat{\pi} = u_0 \hat{\pi}_1 u_1 \dots \hat{\pi}_k u_k$.

Using Lemma 3.5, $\hat{\pi}_i$ is well defined.

Theorem 3.8 *If* $\mathrm{V} = \mathrm{DPerm} \cap [\![x^\omega y x^{\omega+1} = x^{\omega+1} y x^\omega]\!]$ *then*

$$\forall \pi \in \bar{\Omega}_n \mathrm{V}, \quad \mathrm{V} \models \pi = \hat{\pi}.$$

Proof. Let $\pi = u_0 \pi_1 u_1 \dots \pi_k u_k$ with $\pi_i = \pi_i^{\omega+1} = \pi_i^\omega \alpha_{1,i} \dots \alpha_{k_i,i} \pi_i^\omega$ and $\alpha_{s,i} = \alpha_{s,i}^{\omega+1} \in \bar{\Omega}_1 \mathrm{V}$ $(1 \le i \le k,\ 1 \le s \le k_i)$. If $c(\alpha_{s,1}) = c(\alpha_{t,j})$ (which implies $\alpha_{s,1}^\omega = \alpha_{t,j}^\omega$), then

$$\begin{aligned}
\pi &= u_0 \pi_1^\omega \alpha_{s,1} (\alpha_{s,1})^\omega \dots \pi_1^\omega \dots \pi_j^\omega \dots \alpha_{t,j} \dots \pi_j^\omega \dots \\
&= u_0 \pi_1^\omega \alpha_{s,1} (\alpha_{t,j})^\omega \dots \pi_1^\omega \dots \pi_j^\omega \dots (\alpha_{t,j})^{\omega+1} \dots \pi_j^\omega \dots \\
&= u_0 \pi_1^\omega \alpha_{s,1} (\alpha_{t,j})^{\omega+1} \dots \pi_1^\omega \dots \pi_j^\omega \dots \alpha_{t,j}^\omega \dots \pi_j^\omega \dots \\
&= u_0 \pi_1^\omega \alpha_{s,1} \alpha_{t,j} \dots \pi_1^\omega \dots \pi_j^\omega \dots \pi_j^\omega \dots \\
&\ \vdots \ \textit{proceeding from left to right, and eliminating the } (\alpha_{s,i})^\omega, \\
&\quad \textit{using Lemma 3.4.} \\
&= \hat{\pi}. \ \square
\end{aligned}$$

Definition 3.9 For a word u and $k \in \mathbb{N}$, define

$$i_k(u) = \begin{cases} \text{prefix of length } k & \text{if } |u| \ge k \\ u & \text{if } |u| < k. \end{cases}$$

The following lemma is a simple application of the fact that the languages of the form uA^* are recognizable by semigroups in K.

Lemma 3.10 If V is a pseudovariety containing K, $\pi \in \bar{\Omega}_n \mathrm{V}$ and $(u_m)_{m \in \mathbb{N}}$ and $(v_m)_{m \in \mathbb{N}}$ are sequences in $\Omega_n \mathrm{V}$ with limit π, then

$$\forall p \in \mathbb{N}\ \exists k \in \mathbb{N}\ \forall l \ge k,\ i_p(u_k) = i_p(u_l) = i_p(v_l). \ \square$$

Using this lemma we have that the following is well defined.

Definition 3.11 Let V a pseudovariety containing K, $\pi \in \bar{\Omega}_n \mathrm{V}$ and $p \in \mathbb{N}$. Define $i_p(\pi)$ as the limit of the almost constant sequence $(i_p(u_k))_{k \in \mathbb{N}}$, where $(u_k)_{k \in \mathbb{N}}$ is any sequence in $\Omega_n V$ with limit π.

Lemma 3.12 Let V be a pseudovariety such that $K \subseteq V \subseteq DS$. If $\pi, \rho \in \overline{\Omega}_n V$, with $\pi \neq \rho$ then

$$(i_m(\pi))_{m\in\mathbb{N}} = (i_m(\rho))_{m\in\mathbb{N}} \Leftrightarrow \exists \alpha, \delta, \pi_1, \rho_1 \in (\overline{\Omega}_n V)^1 : \begin{cases} \pi = \alpha\delta^\omega \pi_1 \\ \rho = \alpha\delta^\omega \rho_1. \end{cases}$$

Proof. Suppose that $(i_m(\pi))_{m\in\mathbb{N}} = (i_m(\rho))_{m\in\mathbb{N}}$. If $\pi \in \Omega_n V$ then $\pi = \rho$. If $\pi \notin \Omega_n V$, let η be an accumulation point of the sequence $(i_m(\pi))_{m\in\mathbb{N}}$.

As $\eta \notin \Omega_n V$, we have by Theorem 2.7, that η is of the form $u_0\eta_1^{\omega+1}\ldots$. So we conclude that $u_0\eta_1^{\omega+1}$ is a left factor of π and ρ. □

From now on V denotes any pseudovariety satisfying the hypothesis of Theorem 3.1, and W(V) the guess, in that theorem, for $J \vee V$.

Lemma 3.13 Let V be a pseudovariety such that $K \subseteq V \subseteq Perm_{(\infty,2,0)}$. If π and ρ are $\mathcal{J}$-equivalent idempotents of $\overline{\Omega}_n W(V)$, then

$$\pi \mathcal{R} \rho \Leftrightarrow \forall m \in \mathbb{N} \quad i_m(\pi) = i_m(\rho).$$

Proof. If $i_m(\pi) = i_m(\rho)$ for all $m \in \mathbb{N}$, using Lemma 3.12 and its notation we may write π and ρ in the form

$$\pi = \alpha\delta^\omega\pi_1 = (\alpha\delta^\omega\pi_1)^\omega, \quad \rho = \alpha\delta^\omega\rho_1 = (\alpha\delta^\omega\rho_1)^\omega.$$

As $\pi \mathcal{J} \rho$, we have $\pi = \pi\rho\pi$ and $\rho = \rho\pi\rho$. Using the pseudoidentities defining W(V), we obtain

$$\begin{aligned} \pi &= \pi\rho\pi = (\pi\rho)^\omega\pi = (\rho\pi)^{\omega+1} = \rho\pi \\ \rho &= \rho\pi\rho = (\rho\pi)^\omega\rho = (\pi\rho)^{\omega+1} = \pi\rho. \end{aligned}$$

And so $\pi \mathcal{R} \rho$. □

Lemma 3.14 If U is a subpseudovariety of DO satisfying $x^\omega a(st)^\omega = x^\omega a(ts)^\omega$, then, for implicit operations π and ρ on U with the same content,

$$U \models x^\omega a\pi^\omega = x^\omega a\rho^\omega.$$

Proof.

$$\begin{aligned} U \models x^\omega a\pi^\omega &= x^\omega a(\pi^\omega\rho^\omega)^\omega\pi^\omega(\rho^\omega\pi^\omega)^\omega \text{ since } U \subseteq DO \\ &= x^\omega a(\rho^\omega\pi^\omega)^\omega\pi^\omega(\pi^\omega\rho^\omega)^\omega \text{ using } x^\omega a(st)^\omega = x^\omega a(ts)^\omega \\ &= x^\omega a\rho^\omega \text{ since } U \subseteq DO. \ \square \end{aligned}$$

Proof of Theorem 3.1.

i) To simplify the notation we will assume that $K \subseteq V \subseteq Perm_{(\infty,2,0)}$. The other cases are treated similarly.

The proof that $J \vee V \subseteq W(V)$ is merely routine.

For the inverse inclusion, suppose that $\pi,\rho\in\overline{\Omega}_n$ W(V) are such that J, V $\models \pi = \rho$. We wish to prove that $\pi = \rho$. If π is an explicit operation then, as J $\models \pi = \rho$, we have $\pi = \rho$.

Using Theorems 2.7, 2.6 and 3.8, we may suppose that

$$\pi = u_0\pi_1^{\omega+1}u_1\dots\,\pi_k^{\omega+1}u_k \qquad \pi_i = \pi_i^{\omega}\tilde{\pi}_i\pi_i^{\omega}$$
$$\rho = u_0\rho_1^{\omega+1}u_1\dots\,\rho_k^{\omega+1}u_k \qquad \rho_i = \rho_i^{\omega}\tilde{\rho}_i\rho_i^{\omega}$$

with $c(\pi_i) = c(\rho_i)$ and $\tilde{\pi}_i$ and $\tilde{\rho}_i$ of the form

$$\tilde{\pi}_i = \alpha_{1,i}\dots\,\alpha_{s_i,i}$$
$$\tilde{\rho}_i = \beta_{1,i}\dots\,\beta_{s_i,i} \qquad \alpha_{j,i},\beta_{j,i}\in\overline{\Omega}_1 W(V)$$

with $c(\alpha_{j,i}) = c(\beta_{j,i})$.

As V $\models \pi = \rho$ we have W(V)$\cap$G $\models \alpha_{j,i} = \beta_{j,i}$, because in V$\cap$G, which is equal to W(V)$\cap$G, we are dealing with abelian groups and $c(\tilde{\pi}_i)\cap c(\tilde{\pi}_j) = \emptyset$ (for $i\neq j$). Then we conclude, using Lemma 3.3, that

$$V \models \pi = \rho \Rightarrow W(V) \models \tilde{\pi}_i = \tilde{\rho}_i \ \ (1\leq i\leq k). \tag{1}$$

By Lemma 3.12, as K $\models \pi = \rho$, π_1 and ρ_1 are of the form

$$\pi_1 = \alpha_1\delta_1^{\omega}\epsilon_1 \ \ (=(\alpha_1\delta_1^{\omega}\epsilon_1)^{\omega+1})$$
$$\rho_1 = \alpha_1\delta_1^{\omega}\eta_1 \ \ (=(\alpha_1\delta_1^{\omega}\eta_1)^{\omega+1})$$

and, from Lemma 3.13, we deduce that $\pi_1\mathcal{R}\rho_1$, or equivalently

$$\pi_1 = \rho_1^{\omega}\pi_1 \qquad \rho_1 = \pi_1^{\omega}\rho_1. \tag{2}$$

Then, for all i we have

$$\begin{aligned}\pi_i &= \pi_i^{\omega}\pi_i^{\omega}\tilde{\pi}_i\pi_i^{\omega}\\ &= \pi_i^{\omega}\rho_i^{\omega}\tilde{\pi}_i\pi_i^{\omega} \text{ using Lemma 3.14}\\ &= \pi_i^{\omega}\rho_i^{\omega}\tilde{\rho}_i\pi_i^{\omega} \text{ using (1)}\\ &= \pi_i^{\omega}\rho_i\end{aligned} \tag{3}$$

and

$$\begin{aligned}\pi &= u_0\pi_1^{\omega}\rho_1u_1\dots\pi_i^{\omega}\rho_iu_i\dots \text{ using (3)}\\ &= u_0\pi_1^{\omega}\rho_1u_1\dots\rho_i^{\omega}\rho_iu_i\dots \text{ using Lemma 3.14}\\ &= u_0\rho_1u_1\dots\rho_iu_i\dots \text{ using (2)}\\ &= \rho.\end{aligned}$$

ii) This is a consequence of i) since DV = (DV$\cap$Com) for V$\subseteq$Com$\vee$N.

iii) Just use i), Lemma 3.2 and the fact that for any pseudovariety U, DU = D(U$\cap$CS). □

Lemma 3.15

$$\begin{aligned}
\mathbf{Ab}\vee\mathbf{N} &= [\![x^\omega = y^\omega,\ x^\omega yz = x^\omega zy]\!] \\
\mathbf{Ab}\vee\mathbf{K} &= [\![x^\omega y^\omega = x^\omega,\ x^\omega yz = x^\omega zy]\!] \\
\mathbf{Ab}\vee\mathbf{D} &= [\![x^\omega y^\omega = y^\omega,\ yzx^\omega = zyx^\omega]\!] \\
\mathbf{Ab}\vee\mathbf{LI} &= [\![x^\omega y^\omega x^\omega = x^\omega, x^\omega yzt^\omega = x^\omega zyt^\omega]\!].
\end{aligned}$$

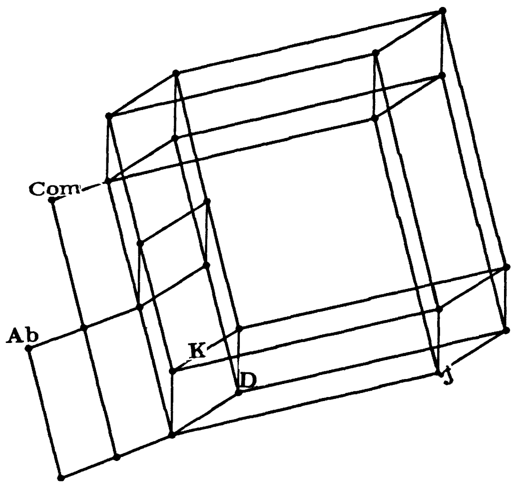
Proof. Let **W** be one of the pseudovarieties **N**, **K**, **D** or **LI** and **U** be my guess for **Ab**$\vee$**W**.

If S$\in$**U** then the Green relation $\mathcal{H}$ is a congruence. Let e be an idempotent of S. Then

$$\begin{aligned}
\varphi : S &\to S/\mathcal{H} \times eSe \\
x &\mapsto ([x], exe)
\end{aligned}$$

is an injective homomorphism. So, as $S/\mathcal{H}\in$**W** and eSe$\in$**Ab**, we have S $\in$ **Ab**$\vee$**W**. □

Nothing that **DAb** = **DCom**, we have proved that he Hasse diagram of the sublattice of the lattice of the permutative pseudovarieties generated by **Com**, **Ab**, **K**, **D** and **J** is the following:



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