Probability Theory and Mathematical Statistics

A WILEY PUBLICATION IN MATHEMATICAL STATISTICS

Probability Theory and Mathematical Statistics

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To Olga and Aleksander

Preface to the English Edition

The opening sentence of this book is "Probability theory is a part of mathematics which is useful in discovering and investigating the regular features of random events." However, even in the not very remote past this sentence would not have been found acceptable (or by any means evident) either by mathematicians or by researchers applying probability theory. It was not until the twenties and thirties of our century that the very nature of probability theory as a branch of mathematics and the relation between the concept of "probability" and that of "frequency" of random events was thoroughly clarified. The reader will find in Section 1.3 an extensive (although, certainly, not exhaustive) list of names of researchers whose contributions to the field of basic concepts of probability theory are important. However, the foremost importance of Laplace's *Théorie Analytique des Probabilités*, von Mises' *Wahrscheinlichkeit, Statistik und Wahrheit*, and Kolmogorov's *Grundbegriffe der Wahrscheinlichkeitsrechnung* should be stressed. With each of these three works, a new period in the history of probability theory was begun. In addition, the work of Steinhaus' school of *independent functions* contributed greatly to the clarification of fundamental concepts of probability theory.

The progress in foundations of probability theory, along with the introduction of the theory of characteristic functions, stimulated the exceedingly fast development of modern probability theory. In the field of limit theorems for sums of random variables, a fairly general theory was developed (Khintchin, Lévy, Kolmogorov, Feller, Gnedenko, and others) for independent random variables, whereas for dependent random variables some important particular results were obtained (Bernstein, Markov, Doeblin, Kolmogorov, and others). Furthermore, the theory of stochastic processes became a mathematically rigorous branch of probability theory (Markov, Khintchin, Lévy, Wiener, Kolmogorov, Feller, Cramér, Doob, and others).

Some ideas on application of probability theory that are now used in mathematical statistics are due to Bayes (estimation theory), Laplace (quality control of drugs), and to Gauss (theory of errors). However, it was not until this century that mathematical statistics grew into a self-contained scientific subject. In order to restrict myself to the names of just a few of the principal persons responsible for this growth, I mention only K. Pearson, R. A. Fisher, J. Neyman, and A. Wald, whose ideas and systematic research contributed so much to the high status of modern mathematical statistics.

At present, the development of the theory of probability and mathematical statistics is going on with extreme intensity. On the one hand, problems in classical probability theory unsolved, as of now, are attracting much attention, whereas, on the other hand, much work is being done in an attempt to obtain highly advanced generalizations of old concepts, particularly by considering probability theory in spaces more general than the finite dimensional Euclidean spaces usually treated. That probability theory is now closely connected with other parts of mathematics is evidenced by the fact that almost immediately after the formulation of the distribution theories by L. Schwartz and Mikusinski their probabilistic counterparts were thoroughly discussed (Gelfand, Itô, Urbanik). Moreover, probability theory and mathematical statistics are no longer simply "customers" of other parts of mathematics. On the contrary, mutual influence and interchange of ideas between probability theory and mathematical statistics and the other areas of mathematics are constantly going on. One example is the relation between analytic number theory and probability theory (Borel, Khintchin, Linnik, Erdös, Kac, Rényi, and others), and another example is the use of the theory of games in mathematical statistics and its stimulating effect on the development of the theory of games itself (v. Neumann, Morgenstern, Wald, Blackwell, Karlin, and others).

Areas of application of probability theory and mathematical statistics are increasing more and more. Statistical methods are now widely used in physics, biology, medicine, economics, industry, agriculture, fisheries, meteorology, and in communications. Statistical methodology has become an important component of scientific reasoning, as well as an integral part of well-organized business and social work.

The development of probability theory and mathematical statistics and their applications is marked by a constantly increasing flood of scientific papers that are published in many journals in many countries.

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Having described briefly the position of modern probability theory and mathematical statistics, I now state the main purpose of this book:

1. To give a systematic introduction to modern probability theory and mathematical statistics.
2. To present an outline of many of the possible applications of these theories, accompanied by descriptive concrete examples.
3. To provide extensive (however, not exhaustive) references of other books and papers, mostly with brief indications as to their contents, thereby giving the reader the opportunity to complete his knowledge of the subjects considered.

Although great care has been taken to make this book mathematically rigorous, the intuitive approach as well as the applicability of the concepts and theorems presented are heavily stressed.

For the most part, the theorems are given with complete proofs. Some proofs, which are either too lengthy or require mathematical knowledge far beyond the scope of this book, were omitted.

The entire text of the book may be read by students with some background in calculus and algebra. However, no advanced knowledge in these fields or a knowledge in measure and integration theory is required. Some necessary advanced concepts (for instance, that of the Stieltjes integral) are presented in the text. Furthermore, this book is provided with a Supplement, in which some basic concepts and theorems of modern measure and integration theory are presented.

Every chapter is followed by "Problems and Complements." A large part of these problems are relatively easy and are to be solved by the reader, with the remaining ones given for information and stimulation.

This book may be used for systematic one-year courses either in probability theory or in mathematical statistics, either for senior undergraduate or graduate students. I have presented parts of the material, covered by this book, in courses at the University of Warsaw (Poland) for nine academic years, from 1951/1952 to 1959/1960, at the Peking University (China) in the Spring term of 1957, and in this country at the University of Washington and at Stanford, Columbia and New York Universities for the last several years.

This book is also suitable for nonmathematicians, as far as concepts, theorems, and methods of application are concerned.

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I started to write this book at the end of 1950. Its first edition (374 pages) was published in Polish in 1954. All copies were sold within a few months. I then prepared the second, revised and extended, Polish edition, which was published in 1958, simultaneously with its German translation. Indications about changes and extensions introduced into the second Polish edition are given in its preface. In comparison with the second Polish edition, the present English one contains many extensions and changes, the most important of which are the following:

Entirely new are the Problems and Complements, the Supplement, and the Sections: 2.7,C, 2.8,C, 3.2,C, 3.6,G, 4.6,B, 6.4,B, 6.4,C, 6.12,D, 6.12,F, 6.15, 8.11, 9.4,B, 9.6,E, 9.9,B, 10.10,B, 10.11,E, 12.6,B, 13.5,E, 13.7,D, 14.2,E, 14.4,D, 15.1,C, 15.3,C, 16.3,D, 16.6, and 17.10,A. Section 8.10 (Stationary processes) is almost entirely new.

Considerably changed or complemented are Sections: 2.5,C, 3.5, 3.6,C, 4.1, 4.2, 5.6,B, 5.7, 5.13,B, 6.2, 6.4,A, 6.5, 6.12,E, 6.12,G, 7.5,B, 8.4,D, 8.8,B, 8.12, 9.1, 9.7, 10.12, 10.13, 12,4,C, 12.4,D, 13.3, 16.2,C.

These changes and extensions have all been made to fulfill more completely the main purpose of this book, as stated previously.

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J. Łukaszewicz, A. M. Rusiecki, and W. Sadowski read the manuscript of the first edition and suggested many improvements. The criticism of E. Marczewski and the reviews by Z. W. Birnbaum and S. Zubrzycki of the first edition were useful in preparing the second edition; also useful were valuable remarks of K. Urbanik, who read the manuscript of the second edition. Numerous remarks and corrections were suggested by J. Wojtyniak, and R. Zasępa (first edition), and by L. Kubik, R. Sulanke, and J. Wloka (second edition). R. Bartoszynski, with the substantial collaboration of Mrs H. Infeld, translated the book from the Polish. J. Karush made valuable comments about the language. Miss D. Garbose did the editorial work. B. Eisenberg assisted me in the reading of the proofs. My sincere thanks go to all these people.

New York
October, 1962

MAREK FISZ

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PART 1

Probability Theory

CHAPTER 1

Random Events

1.1 PRELIMINARY REMARKS

A Probability theory is a part of mathematics which is useful in discovering and investigating the regular features of random events. The following examples show what is ordinarily understood by the term random event.

Example 1.1.1. Let us toss a symmetric coin. The result may be either a head or a tail. For any one throw, we cannot predict the result, although it is obvious that it is determined by definite causes. Among them are the initial velocity of the coin, the initial angle of throw, and the smoothness of the table on which the coin falls. However, since we cannot control all these parameters, we cannot predetermine the result of any particular toss. Thus the result of a coin tossing, head or tail, is a random event.

Example 1.1.2. Suppose that we observe the average monthly temperature at a definite place and for a definite month, for instance, for January in Warsaw.[1] This average depends on many causes such as the humidity and the direction and strength of the wind. The effect of these causes changes year by year. Hence Warsaw's average temperature in January is not always the same. Here we can determine the causes for a given average temperature, but often we cannot determine the reasons for the causes themselves. As a result, we are not able to predict with a sufficient degree of accuracy what the average temperature for a certain January will be. Thus we refer to it as a random event.

B It might seem that there is no regularity in the examples given. But if the number of observations is large, that is, if we deal with a mass phenomenon, some regularity appears.

Let us return to example 1.1.1. We cannot predict the result of any particular toss, but if we perform a long series of tossings, we notice that the number of times heads occur is approximately equal to the number of times tails appear. Let n denote the number of all our tosses and m the number of times heads appear. The fraction m/n is called the

[1] See example 12.5.1.

frequency of appearance of heads. The frequency of appearance of tails is given by the fraction $(n - m)/n$. Experience shows that if n is sufficiently large, thus if the tossings may be considered as a mass phenomenon, the fractions m/n and $(n - m)/n$ differ little; hence each of them is approximately $\frac{1}{2}$. This regularity has been noticed by many investigators who have performed a long series of coin tossings. Buffon

TABLE 1.1.1

FREQUENCY OF BIRTHS OF BOYS AND GIRLS

Year of Birth	Number of Births		Total Number of Births	Frequency of Births	
	Boys m	Girls f	$m + f$	Boys p_1	Girls p_2
1927	496,544	462,189	958,733	0.518	0.482
1928	513,654	477,339	990,993	0.518	0.482
1929	514,765	479,336	994,101	0.518	0.482
1930	528,072	494,739	1,022,811	0.516	0.484
1931	496,986	467,587	964,573	0.515	0.485
1932	482,431	452,232	934,663	0.516	0.484
Total	3,032,452	2,833,422	5,865,874	0.517	0.483

tossed a coin 4040 times, and obtained heads 2048 times; hence the ratio of heads was $m/n = 0.50693$. In 24,000 tosses, K. Pearson obtained a frequency of heads equal to 0.5005. We can see quite clearly that the observed frequencies oscillate about the number 0.5.

As a result of long observation, we can also notice certain regularities in example 1.1.2. We investigate this more closely in example 12.5.1.

Example 1.1.3. We cannot predict the sex of a newborn baby in any particular case. We treat this phenomenon as a random event. But if we observe a large number of births, that is, if we deal with a mass phenomenon, we are able to predict with considerable accuracy what will be the percentages of boys and girls among all newborn babies. Let us consider the number of births of boys and girls in Poland in the years 1927 to 1932. The data are presented in Table 1.1.1. In this table m and f denote respectively the number of births of boys and girls in particular years. Denote the frequencies of births by p_1 and p_2, respectively; then

$$p_1 = \frac{m}{m + f}, \qquad p_2 = \frac{f}{m + f}.$$

One can see that the values of p_1 oscillate about the number 0.517, and the values of p_2 oscillate about the number 0.483.