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A Supplement to the Usual Treatises

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ON RIEMANN'S THEORY

OF

ALGEBRAIC FUNCTIONS

AND THEIR

INTEGRALS.

A SUPPLEMENT TO THE USUAL TREATISES.

BY

FELIX KLEIN.

TRANSLATED FROM THE GERMAN, WITH THE AUTHOR'S PERMISSION,

BY

FRANCES HARDCASTLE,

GIRTON COLLEGE, CAMBRIDGE.

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TRANSLATOR'S NOTE.

The aim of this translation is to reproduce, as far as possible, the ideas and style of the original in idiomatic English, rather than to give a literal rendering of its contents. Even the verbal deviations, however, are few in number. So little has been written in English on the subject that a standard set of technical terms as yet hardly exists. Where there was any choice between equivalent words, I have followed the usage of Dr Forsyth in his recently published work on the Theory of Functions. A Glossary of the principal technical terms is appended, giving the original German word together with the English adopted in the text.

Prof. Klein had originally intended to revise the proofs, but owing to his absence in America he kindly waived his right to do so, in order not to delay the publication. The proofs have therefore not been submitted to him, though it was with considerable reluctance that I determined to publish without this final revision.

My thanks are due to Miss C. A. Scott, D.Sc., Professor of Mathematics in Bryn Mawr College, for many valuable suggestions in difficult passages and for her interest in the progress of the translation, and also for help in the reading of the proof-sheets. I must also express my thanks to Mr James Harkness, M.A., Associate Professor of Mathematics in Bryn Mawr College, for helpful advice given from time to time; and to Miss P. G. Fawcett, of Newnham College, Cambridge, for reading over in manuscript the earlier parts which deal more especially

with the subject of Electricity.

FRANCES HARDCASTLE.

Bryn Mawr College,
Pennsylvania,
June 1, 1893.

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PREFACE.

The pamphlet which I here lay before the public, has grown from lectures delivered during the past year,[*] in which, among other objects, I had in view a presentation of Riemann's theory of algebraic functions and their integrals.[†] Lectures on higher mathematics offer peculiar difficulties; with the best will of the lecturer they ultimately fulfil a very modest purpose. Being usually intended to give a *systematic* development of the subject, they are either confined to the elements or are lost amid details. I thought it well in this case, as previously in others, to adopt the opposite course. I assumed that the ordinary presentation, as given in text-books on the elements of Riemann's theory, was known; moreover, when particular points required to be more fully dealt with, I referred to the fundamental monographs. But to compensate for this, I devoted great care to the presentation of the *true train of thought*, and endeavoured to obtain a *general view* of the scope and efficiency of the methods. I believe I have frequently obtained good results by these means, though, of course, only with a gifted audience; experience will show whether this pamphlet,

[*] *Theory of Functions treated geometrically.* Part I, Winter-semester 1880–81, Part II, Summer-semester 1881.

[†] I denote thus the contents of the investigations with which Riemann was concerned in the first part of his *Theory of the Abelian Functions*. The theory of the Θ-functions, as developed in the second part of the same treatise, is in the first place, as we know, of an essentially different character, and is excluded from the following presentation as it was from my course of lectures.

based on the same principles, will prove equally useful.

A presentation of the kind attempted is necessarily very subjective, and the more so in the case of Riemann's theory, since but scanty material for the purpose is to be found explicitly given in Riemann's papers. I am not sure that I should ever have reached a well-defined conception of the whole subject, had not Herr Prym, many years ago (1874), in the course of an opportune conversation, made me a communication which has increased in importance to me the longer I have thought over the matter. He told me that *Riemann's surfaces originally are not necessarily many-sheeted surfaces over the plane, but that, on the contrary, complex functions of position can be studied on arbitrarily given curved surfaces in exactly the same way as on the surfaces over the plane.* The following presentation will sufficiently show how valuable this remark has been to me. In natural combination with this there are certain physical considerations which have been lately developed, although restricted to simpler cases, from various points of view.* I have not hesitated to take these physical conceptions as the starting-point of my presentation. Riemann, as we know, used Dirichlet's Principle in their place in his writings. But I have no doubt that he started from precisely those physical problems, and then, in order to give what was physically evident the support of mathematical reasoning, he afterwards substituted Dirichlet's Principle. Anyone who clearly

*Cf. C. Neumann, Math. Ann., t. x., pp. 569–571. Kirchhoff, *Berl. Monatsber.*, 1875, pp. 487–497. Töpler, *Pogg. Ann.*, t. clx., pp. 375–388.

understands the conditions under which Riemann worked in Göttingen, anyone who has followed Riemann's speculations as they have come down to us, partly in fragments,* will, I think, share my opinion.—However that may be, the physical method seemed the true one for my purpose. For it is well known that Dirichlet's Principle is not sufficient for the actual foundation of the theorems to be established; moreover, the heuristic element, which to me was all-important, is brought out far more prominently by the physical method. Hence the constant introduction of intuitive considerations, where a proof by analysis would not have been difficult and might have been simpler, hence also the repeated illustration of general results by examples and figures.

In this connection I must not omit to mention an important restriction to which I have adhered in the following pages. We all know the circuitous and difficult considerations by which, of late years, part at least of those theorems of Riemann which are here dealt with have been proved in a reliable manner.† These considerations are entirely neglected in what follows and I thus forego the use of any except intuitive bases for the the-

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* *Ges. Werke*, pp. 494 *et seq.*

† Compare in particular the investigations on this subject by C. Neumann and Schwarz. The general case of *closed* surfaces (which is the most important for us in what follows) is indeed, as yet, nowhere explicitly and completely dealt with. Herr Schwarz contents himself with a few indications with respect to these surfaces (*Berl. Monatsber.*, 1870, pp. 767 *et seq.*) and Herr C. Neumann only considers those cases in which functions are to be determined by means of known values on the boundary.

orems to be enunciated. In fact such proofs must in no way be mixed up with the sequence of thought I have attempted to preserve; otherwise the result is a presentation unsatisfactory from all points of view. But they should assuredly follow after, and I hope, when opportunity offers, to complete in this sense the present pamphlet.

For the rest, the scope and limits of my presentation speak for themselves. The frequent use of my friends' publications and of my own on kindred subjects had a secondary purpose important to me for personal reasons: I wished to give my audience a guide, to help them to find for themselves the reciprocal connections among these papers, and their position with respect to the general conception put forth in these pages. As for the *new* problems which offer themselves in great number, I have only allowed myself to investigate them as far as seemed consistent with the general aim of this pamphlet. Nevertheless I should like to draw attention to the theorems on the conformal representation of arbitrary surfaces which I have worked out in the last Part; I followed these out the more readily that Riemann makes a remarkable statement about this subject at the end of his Dissertation.

One more remark in conclusion to obviate a misunderstanding which might otherwise arise from the foregoing words. Although I have attempted, in the case of algebraic functions and their integrals, to follow the original chain of ideas which I assumed to be Riemann's, I by no means include the whole of what he intended in the theory of functions. The said functions were for him an example only, in the treatment of which, it is true, he was particularly fortunate. Inasmuch as he wished to include all possible functions of complex variables, he had in mind far more general methods of determination than those we employ in the following pages; methods of determination in which physical analogy, here deemed a sufficient basis, fails us. Compare, in this connection, § 19 of his Dissertation, compare also his work on the hypergeometrical series.—With reference to this, I must explain that I have no wish to draw aside from these more general considerations by giving a presentation of a special part, complete in itself. My innermost conviction rather is that they are destined to play, in the developments of the modern Theory of Functions, an important and prominent part.

Borkum,
Oct. 7, 1881.

PART I.

Introductory Remarks.

§ 1. *Steady Streamings in the Plane as an Interpretation of the Functions of $x + iy$.*

The physical interpretation of those functions of $x + iy$ which are dealt with in the following pages is well known.* The principles on which it is based are here indicated, solely for completeness.

Let $w = u + iv$, $z = x + iy$, $w = f(z)$. Then we have, primarily,

$$(1) \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

and hence

$$(2) \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

and also, for v,

$$(3) \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

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*In particular, reference should be made to Maxwell's *Treatise on Electricity and Magnetism* (Cambridge, 1873). So far as the intuitive treatment of the subject is concerned, his point of view is exactly that adopted in the text.

In these equations we take u to be the *velocity-potential*, so that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ are the components of the velocity of a fluid moving parallel to the xy plane. We may either suppose this fluid to be contained between two planes, parallel to the xy plane, or we may imagine it to be itself an infinitely thin homogeneous sheet extending over this plane. Then equation (2)—and this is the chief point in the physical interpretation—shows that the streaming is *steady*. The curves $u =$ const. are called the *equipotential curves*, while the curves $v =$ const., which, by (1), are orthogonal to the first system, are the *stream-lines*. For the purposes of this interpretation it is of course indifferent of what nature we may imagine the fluid to be, but for many reasons it will be convenient to identify it here with the *electric fluid*; u is then proportional to the electrostatic potential which gives rise to the streaming, and the apparatus of experimental physics provide sufficient means for the production of many interesting systems of streamings.

Moreover, if we increase u throughout by a constant the streaming itself remains unchanged, since the differential coefficients $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ alone appear explicitly; this is also true of v. Hence the function $u + iv$, whose physical interpretation is in question, is thus determined only to an additive constant près, a fact which requires to be carefully observed in what follows.

Further, we may observe that equations (1)–(3) remain unaltered if we replace u by v, and v by $-u$. Corresponding to this we get a second system of streamings in which v is the velocity-potential and the curves $u =$ const. are the stream-lines; in the sense explained above this represents the function $v - iu$. It is often of use to consider this new streaming as well as the original one in which u was the velocity-potential; we shall speak of it, for brevity, as the *conjugate* streaming. It is true that the name is somewhat inaccurate, since u bears the same relation to v, as v does to $-u$, but it is sufficiently intelligible for our purpose.

The differential equations (1)–(3), and hence also the whole preceding discussion, apply in the first place solely to that portion of the plane (otherwise an arbitrary portion) in which $u+iv$ is uniform and in which neither $u+iv$ nor its differential coefficients become infinite. In order then that the corresponding physical conditions maybe clearly comprehended, a region of this kind must be marked off and then by suitable appliances on the boundary the steady motion within its limits must be preserved.

In a bounded region of this description points z_0 at which the differential coefficient $\dfrac{\partial w}{\partial z}$ vanishes call for special attention To be perfectly general, I will assume at once that $\dfrac{\partial^2 w}{\partial z^2}$, $\dfrac{\partial^3 w}{\partial z^3}$, ..., up to $\dfrac{\partial^\alpha w}{\partial z^\alpha}$ are all zero as well. To determine the course of the equipotential curves, or of the stream-lines in the vicinity of such a point, let w be expanded in a series of ascending powers of $z - z_0$; in this series, the term immediately after the constant term is the term in $(z - z_0)^{\alpha+1}$. Transforming to polar-coordinates we obtain the following result: *at the point z_0, $\alpha+1$ curves $u =$ const. intersect at equal angles, while the same number of curves $v =$ const. are the bisectors of these angles.* In consequence of this property I call such a point a *cross-point*, and moreover a *cross-point of multiplicity α.*

The following figure (which is of course only diagrammatic) illustrates this for $\alpha = 2$, and explains, in particular, how a cross-point makes its appearance in the orthogonal system

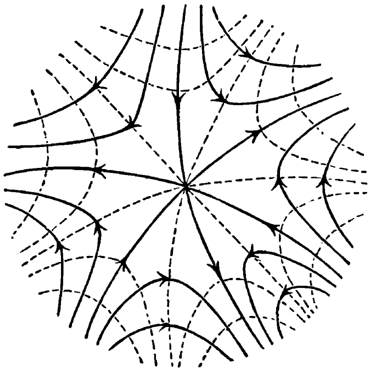

Fig. 1.

formed by the curves $u =$ const., $v =$ const.

The stream-lines $v =$ const. are the heavy lines in the figure and the direction of motion in each is indicated by an arrow; the equipotential curves are given by dotted lines. We see how the fluid flows in towards the cross-point from three directions, and flows out again in three other directions, this being possible because the velocity of the streaming is zero at the cross-point, or, as we may say, by analogy with known occurrences, because the fluid is at a standstill, the expression for the velocity being $\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}$.

Further, it is useful to consider a cross-point of multiplicity α *as the limiting case of* α *simple cross-points*. The analytical treatment shows this to be permissible. For at an α-ple cross-point the equation $\frac{\partial w}{\partial z} = 0$ has an α-ple root and this is caused, as we know, by the coalescence of α simple roots. The following figures sufficiently explain this view:

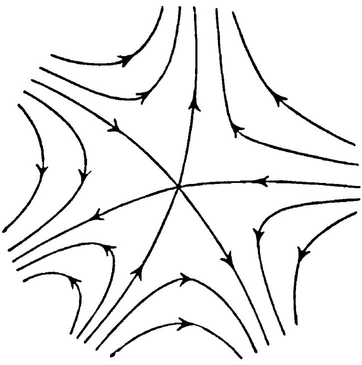

Fig. 2.

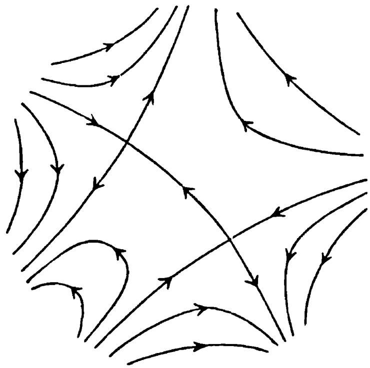

Fig. 3.

For simplicity, I have here drawn the stream-lines only. On the left we have the same cross-point of multiplicity two as in Fig. 1; on the right we have a streaming with two simple cross-points close together. It is at once evident that the one figure is produced by continuous changes from the other.

Throughout the foregoing discussion it has been tacitly assumed that the region in question does not extend to infinity. It is true that no fundamental difficulties present themselves when we take the point $z = \infty$ into account exactly as we take any other point $z = z_0$; instead of the expansion in ascending powers of $z - z_0$, we obtain, by known methods, an expansion in ascending powers of $\dfrac{1}{z}$; there is an α-ple cross-point at $z = \infty$ when the term immediately following the constant term in this expansion is the term in $\left(\dfrac{1}{z}\right)^{\alpha+1}$. But we need dwell no further on the geometrical relations corresponding to a streaming of this kind, for the separate treatment of $z = \infty$, which here presents itself, will be obviated once and for all by a method to be explained shortly, and for this reason the point $z = \infty$ will be left out of consideration in the following sections (§§ 2–4), although, if a complete treatment were desired, it ought to be specially mentioned.

§ 2. *Consideration of the Infinities of $w = f(z)$.*

We now further include in this region points z_0 at which $w = f(z)$ becomes infinite. But, since we are about to consider only a special class of functions, we restrict ourselves in this direction by the following condition, viz.: *the differential coefficient $\dfrac{\partial w}{\partial z}$ must have no essential singularities*, or, in other words, *w is to be infinite only in the same manner as an*