CHAPTER 4

LINEAR OPERATORS

The initiators of the spectral theory of linear operators have so strongly associated, in the mathematical consciousness, the term "self-adjoint" with the term "operator" that up to now, when mathematicians think of a linear transformation of a sufficiently general form, many of them prefer to say "NON-self-adjoint," which, obviously, reminds us of the anecdotal "... but it is only you I love." The problems which constitute this chapter are not of this kind. Almost all of them belong to the spectral theory which tends to be mixed with complex analysis and in any case it borrows from the latter much more than the Liouville and Stone—Weierstrass theorems, to which the classical approach has been basically restricted. The mentioned intermixing constitutes probably the most characteristic feature of the present state of the theory, and the origin of this interpenetration has to be sought in the works of the late thirties (H. Wold, A. I. Plesner).

The new spectral theory is oriented from the very beginning towards the creation of suitable *functional models* while for the classical analysis such a model often concludes the investigation. The majority of the problems in this chapter are related to the multiplication operator $f \to zf$, whose restrictions and projections onto appropriate subspaces give the abovementioned models. One of the popular models is described in terms of the so-called characteristic function of the operator (the Sz.-Nagy—Foias model and its generalizations). In this case the problems of the spectral theory are reduced to the investigation of the boundary properties of vector-valued functions of the Nevanlinna class. The questions raised in Secs. 5.4-10.4 characterize in a sufficiently precise manner the state of the subject: almost all the achievements of the theory of Hardy classes H^p have been already used by spectral analysis (the results on free interpolation, the thin multiplicative structure of the H^p-functions, the corona theorem, etc.) and the problems which arise require either new efforts in the traditional key of the H^p-theory (Secs. 8.4, 6.4) or the creation of a special noncommutative intuition which would allow us to discover, by means of formally invariant formulations (relative to dimension), a content in the operator-valued function theory, basically different from the one in the scalar case. As brilliant examples we mention the Halmos—Lax theorem, describing the invariant subspaces of the multiple shift operator (its deciphering in a very special situation is considered in Sec. 5.4) and the problem 8.4. In the last case the simplicity and economy of the formulation conceal in an especially good manner the interesting realization of the problem; Sec. 8.4 contains both the fundamental question from 7.4 and the following generalization of the "corona problem": if f_n, $n \in \mathbb{N}$, are analytic functions in $\mathbb{D}$ and $0 < \inf\{\sum_{n\geq 1} |f_n(\zeta)|^2 : \zeta \in \mathbb{D}\} \leq \sup\{\sum_{n\geq 1} |f_n(\zeta)|^2 : \zeta \in \mathbb{D}\} < +\infty$, then do there exist functions g_n, $n \in \mathbb{N}$, analytic in $\mathbb{D}$, for which $\sum_{n\geq 1} f_n g_n = \mathbb{1}$, $\sup\{\sum_{n\geq 1} |g_n(\zeta)|^2 : \zeta \in \mathbb{D}\} < +\infty$? Other problems (3.4, 6.4, 7.4, 13.4) of the vectorial theory of functions, placed in this chapter, admit similar interpretations. We mention only the repeated treatment of the problems of the linear similarity of operators (7.4, 9.4, 11.4, 13.4) which apparently has arisen as a consequence of searching for systems of unitary invariants.

Some other models are touched upon in Secs. 2.4 and 3.4, where (on a completely different material!) one elucidates the role of the peak sets for describing the spectral character of special operators. It is no wonder that these problems are closely related to Chaps. 7 and 10 and, basically, could have been placed in these chapters. The arbitrariness of the division into chapters is demonstrated also by other sections: The problem 1.4 is related to Chap. 6, the content of Secs. 8.2, 7.6, 8.6, and 1.9 is related to the problems of the theory of Toeplitz operators (compare with 11.4, 12.4) and, for example, an entire series of problems of Chap. 5 (6.5, 7.5, 8.5, 2.5, and 3.5) have as a matter of fact an operator origin and meaning. Here it is appropriate to mention also the connection of the same question 8.4 with the estimates of the "truncating" functions g_n ($n \in \mathbb{N}$), obtained within the framework of the theory of ideals (see the well-known works of L. Hörmander, J. Kelleher, B. Taylor, and J.-P. Ferrier in the bibliography to the problems of Chap. 5).

The descriptive theory of operators is represented by Secs. 1.4, 14.4, and 15.4, so that the reader is justified in not knowing, for example, about the unsolved problem regarding the existence of nontrivial invariant subspaces for linear operators in a Hilbert space. Besides, this problem can be "computed" from the constructions of Sec. 1.4, while other problems in the same direction will be better presented in some other collection.

1.4. IS THE UNIFORM ALGEBRAIC APPROXIMATION OF THE MULTIPLICATION AND CONVOLUTION OPERATORS POSSIBLE?†

0. A New Definition. A family $\mathfrak{A}=\{A_\omega:\omega\in\Omega\}$ of bounded operators in a Hilbert space H is said to be uniformly algebraically approximable or, briefly, *approximable*, if $\forall\varepsilon>0\ \exists A_{\omega,\varepsilon}$, $\omega\in\Omega$: a) $\sup_{\omega\in\Omega}\|A_{\omega,\varepsilon}-A_\omega\|<\varepsilon$, b) the *-algebra (i.e., the algebra containing with B also the operator B*) $\mathfrak{A}_\varepsilon$, spanned by $A_{\omega,\varepsilon},\omega\in\Omega$, is finite-dimensional.‡ In particular, the operator A is approximable if the one-element family $\mathfrak{A}=\{A\}$ is approximable; in this case, also the pair {Re A, Im A} is approximable. The function $H(\varepsilon,\mathfrak{A})\equiv\log_2\dim\mathfrak{A}^\circ_\varepsilon$, $\varepsilon>0$, where $\mathfrak{A}^\circ_\varepsilon$ is the algebra of minimal dimension, occurring in the definition of approximability, is called the *entropy growth* of the family $\mathfrak{A}$.

1. Fundamental Problem. Give suitable tests for the approximability of a family of operators, in particular, for one (non-self-adjoint) operator; construct a calculus for approximable families. Concrete analytic problems are in Sec. 5.

2. Known Approximable Families. a) $\mathfrak{A}=\{A\}$, $A=A^*$; indeed, we set $A_\varepsilon=\sum_{i=1}^n\lambda_i(P_{\lambda_i}-P_{\lambda_{i-1}})$, where $\{\lambda_i\}_{i=1}^n$ is an ε-net of the spectrum, $\{P_\lambda\}$ is the spectral expansion of the self-adjoint operator A; in this case, $H(\varepsilon,\mathfrak{A})$ is the usual ε-entropy of the spectrum Spec A as a compactum in $\mathbb{R}$. b) $\mathfrak{A}=\{A_1,\ldots,A_n\}$, $A_iA_j=A_jA_i$, $A_i^*=A_i$, $i,j=1,\ldots,n$. The operators $A_{i,\varepsilon}$ are constructed in the same way; $H(\varepsilon,\mathfrak{A})$ is again the ε-entropy of the joint spectrum as a compactum in $\mathbb{R}^n$. c) The same is valid for a finite family of commuting normal operators. d) $\mathfrak{A}$ is a finite (or compact) subset of $\mathfrak{L}C(H)$, where $\mathfrak{L}C(H)$ is the space of all compact operators in H; here the operators A_ε can be taken to be finite-dimensional. e) $\mathfrak{A}=\{\mathfrak{A}', B_1\ldots B_n\}$, where $\mathfrak{A}'$ is approximable, while $B_i\in\mathfrak{L}C(H)$, $i=1,\ldots,n$; therefore, an operator with a completely continuous imaginary part is approximable. f) $\mathfrak{A}=\{A_i=\int_X\oplus A_i^x\,dx : i=1,\ldots,n\}$ (matrix functions); here $H=\int_X\oplus H_x\,dx$, $\dim H_x=n<\infty$. g) $\mathfrak{A}=\{P_1,P_2\}$, where $P_{1,2}$ are orthogonal projections; this is a special case of the example f) since there exists an expansion $H=\int_X\oplus\mathbb{C}^2dx$ and $P_{1,2}$ has the form $P_i=\int_X\oplus P_i^x\,dx$ (see, e.g., [2]). h) $\mathfrak{A}=\{U\}$, U being a one-sided shift. If $\mathfrak{B}$ is the C*-algebra generated by U, then it is easy to see (see, e.g., [3]) that $\mathfrak{B}\supset\mathfrak{L}C(H)$ and $\mathfrak{B}/\mathfrak{L}C(H)=C(\mathbb{T})$. Therefore, U is approximable in the Calkin algebra. i) In [4] one proves actually that $\mathfrak{A}=\{A_1,\ldots,A_n\}$ is approximable if A_i are commuting quasinilpotent operators (see [5]).

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‡We do not require that the identity of the algebra $\mathfrak{A}_\varepsilon$ be the identity operator in H, desiring to cover with our definition the completely continuous operators; if one assumes that the identity in $\mathfrak{A}_\varepsilon$ is the identity operator I of H, then the algebra $\mathfrak{A}_\varepsilon$ does not contain completely continuous operators and defines a decomposition $H=H_1^\varepsilon\otimes H_2^\varepsilon$, $\dim H_2^\varepsilon<\infty$, with $A_{\omega,\varepsilon}=I_1\otimes a_{\omega,\varepsilon}$; $a_{\omega,\varepsilon}\in\mathfrak{L}(H_2^\varepsilon)$, $\omega\in\Omega$. In general, in these questions it is convenient that all the occurring algebras be factorized with respect to the ideal of the completely continuous operators and, thus, to consider the approximability in the Calkin algebra $\mathfrak{L}(H)/\mathfrak{L}C(H)$ [see Example h) of Sec. 2]. For definitions in the theory of C*-algebras, see [1].

3. Known Nonapproximable Families. a) THEOREM. If the family $\mathfrak{A}=\{U_i: i=1,\dots,n\}$, where U_i are unitary operators, is approximable, then the group generated by the family $\{U_i\}_{i=1}^n$ is amenable (i.e., has an invariant mean). See, for example, [6]. b) COROLLARY. If $\mathfrak{A}=\{P_i\}_{i=1}^n$, $n > 2$, is a family of orthogonal projections of general position, then $\mathfrak{A}$ is not approximable. (Indeed, we set $U_i = 2P_i - I$, $i = 1,\dots,n$, $U_i = U_i^* = U_i^{-1}$; $\{U_i\}_{i=1}^n$ generate the free product of n groups $\mathbb{Z}_2$, which is not amenable for $n > 2$). c) The more so, two (or more) unitary or self-adjoint operators in the general position do not form an approximable pair. Therefore, one non-self-adjoint operator of general position is not approximable. Approximability imposes certain conditions on the character of the invariant subspaces (see the footnote in Sec. 1). d) If $\mathfrak{A}=\{U_1,\dots,U_n\}$, where U_i are partial isometries, connected by the relation $\sum_{i=1}^{n} U_i U_i^* = I$, $n\geqslant 2$, then, as shown in [7], $\mathfrak{A}$ is not approximable, although the algebra generated by the family $\mathfrak{A}$ is amenable (see [8]). e) An algebra spanned on an approximable family $\mathfrak{A}$ is a subalgebra of the inductive limit of C*-algebras of type I. Therefore, it is amenable as a C*-algebra [7]. However, the class of such algebras is already the class of amenable algebras; see d). If an approximable family $\mathfrak{A}$ generates a factor in H, then, clearly, it is hyperfinite [6]. All this gives necessary conditions for approximability.

4. Justification of the Formulation. Many families of operators, occurring within the frames of an analysis problem, are approximable, since the operators considered simultaneously in applications cannot be "too" noncommutative (see [9, 10], problems of perturbation theory, the theory of the representation of several noncommutative groups, etc.). The approximable families are the simplest after the finite-dimensional, noncommutative families.

On the other hand, for such a family one can construct an operational calculus on the basis of the theory of the usual matrices. Indeed, the functions of the noncommuting terms of the approximated family can be determined as uniform limits of functions of matrices; therefore, it seems plausible that for such families one can construct correctly a functional calculus, symbols, various models and canonical forms, and one can investigate the lattices of invariant subspaces, etc. In particular, if A is a non-self-adjoint approximable operator and its spectrum does not consist of a single point, then, apparently, one can show that A has nontrivial invariant subspaces.

It is known that weak approximation (which takes place for any finite families) is not sufficient for the construction of a meaningful functional calculus of noncommuting operators. However, it is possible to have also other concepts of approximation, intermediary between uniform and weak approximation. (See, for example, the definition of a pseudofinite family in [6].)

5. More Concrete Problems. In the most vivid form, the raised topic is concentrated in the following questions. The author avoids expressing them in the form of a definite conjecture.

a) Is the shift U (on a group) and the multiplication V (by a function) an approximable pair of operators? More exactly, let G be a locally compact Abelian group, $g \in G$, let $\varphi\in\hat{G}$ be a character of G, $(Uf)(x)=f(gx)$, $Vf=\varphi f$, $f\in L^2_m(G)$. For example, 1) $G=\mathbb{Z}$, $(Uf)_n=f_{n+1}$, $(Vf)_n=e^{i\alpha n}f_n$, $\alpha\in\mathbb{T}$, $f=\{f_n\}\in\ell^2$; 2) $G=\mathbb{T}$, $f\in L^2_m(\mathbb{T})$, $(Uf)(e^{i\theta})=f(e^{i(\theta+\alpha)})$, $Vf(e^{i\theta})=e^{i\theta}f(e^{i\theta})$; 3) $G=\hat{G}=\mathbb{R}$, $(Vf)(x)=e^{itx}f(x)$, $(Uf)(x)=f(x+\tau)$, $t,\tau\in\mathbb{R}$, $f\in L^2(\mathbb{R})$, etc.

The elucidation of the approximability of the pair {U, V} requires a detailed investigation, useful in its own right, of the mutual disposition of the spectral subspaces of these operators. Under one of the approaches the problem consists in the following: We consider the partition of the circumference $\mathbb{T}=\bigcup_{i=1}^{n} l_i$, l_i are arcs $i = 1,\dots,n$, $L^2(\mathbb{T})=\sum_{i=1}^{n}\oplus L^2_{l_i}$. Let $H = H_{\alpha,\varepsilon}$, $\varepsilon > 0$, be the space of functions for which the only nonzero Fourier coefficients are those whose indices n satisfy the condition: $|\{n\alpha\}| < \varepsilon$; $\{\cdot\}$ is the fractional part, α is irrational. The question consists in finding the disposition of the subspaces $H_{\alpha,\varepsilon}$ and $L^2_{l_i}$ in $L^2_m(\mathbb{T})$, i.e., their mutual projections, stationary angles, etc. Since U and V satisfy the equation $(G = \mathbb{T})$ $VUV^{-1}U^{-1} = e^{i\alpha}I$ (Heisenberg's commutation relation), the question can be posed also in the following form: Can this matrix equation be solved approximately in the uniform topology?

The group shift can be replaced by a more general dynamical system with an invariant measure (X, T, μ) and one can consider the operators $(U_T f)(x) = f(Tx)$ and $V_\varphi f = \varphi f$, $\varphi \in L^\infty$ in $L^2_\mu(X)$. The approximability of the pair $\{U_T, V_\varphi\}$ depends essentially on the (not only spectral) properties of the dynamical system. The author is not aware of any investigation of this topic. We note that the numerous existent approximations in ergodic theory are of no help here since one can easily see that the uniform operator topology on the group of unitary operators, generated by the dynamical system, is discrete. We also note that in the case of a positive answer certain singular operators, the operators of Bishop–Halmos type [11], etc. would turn out to be approximable. This would allow us to give a direct proof of the existence of invariant subspaces (see Sec. 4).

b) Let A be a contraction in H; can one give suitable approximability criteria in terms of the dilatation of A or of the characteristic functions?

c) Let A be the integral operator $(Af)(x) = \int_X K(x,y) f(y)\, d\mu(y)$, $f \in L^2_\mu(X)$. Find approximability criteria in terms of this kernel. Especially interesting are the nonnegative kernels K $(K \geqslant 0)$.

d) For which countable solvable groups G of rank 2, is the regular representation by (unitary) operators in $l^2(G)$ an approximable family? For which general locally compact groups is this so?

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†The author has been informed about the existence of [4] by M. I. Zakharevich and about [7] by A. A. Lodkin.