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# The Theory of Lie Superalgebras

An Introduction

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*To Irene*

# PREFACE

The theory of Lie superalgebras (or, as they are also called, $Z_2$-graded Lie algebras) has undergone a remarkable evolution during the last few years. At present the most important result in the theory seems to be the classification by V.G. Kac of the finite-dimensional simple Lie superalgebras over an algebraically closed field of characteristic zero. Our main objective is to give a self-contained and detailed presentation of this classification. Thus we shall not presuppose any knowledge of the theory of Lie superalgebras, however, we assume that the reader is familiar with the standard theory of Lie algebras.

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Manfred Scheunert

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## INTRODUCTION

During the last few years the theory of Lie superalgebras has seen a remarkable evolution, both in mathematics and physics. The reader who is interested in the historical background is referred to the review by Corwin, Ne'eman and Sternberg [1] which presents the subject as it was known in 1974. As a recent survey of the physical applications we mention the article by Fayet and Ferrara [2]. Both of these works contain an extensive bibliography. The most comprehensive description of the mathematical theory of Lie superalgebras is due to Kac [3] (a sketch of this article has been given in [4]).

The present work, too, is concerned with the mathematical side of the subject. Our main intent is to give a self-contained and detailed presentation of the classification of all finite-dimensional simple Lie superalgebras over an algebraically closed field of characteristic zero, a classification which has been obtained by Kac [3]. The difficulty lies in the fact that the Killing form of a simple Lie superalgebra may be equal to zero. Thus the techniques which are commonly used in the classification of semi-simple Lie algebras are not applicable here. But even if one is willing to assume in addition that the Killing form is non-degenerate one still has to cope with the problem that normally this form induces a non-definite bilinear form on the real vector space spanned by the roots. An investigation along these lines has been carried out by Kaplansky [5] (see also [3]).

Having in mind the classification of all simple Lie superalgebras we have to look for different techniques and only use the Killing form where it is already known to be non-degenerate (or else to exploit the very fact that the Killing form is equal to zero).

Let us describe the approach which will be chosen in the present work. To do so we have to be a little more explicit. A Lie superalgebra L is a $Z_2$-graded algebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ ; $Z_2 = \{\bar{0},\bar{1}\}$, whose defining commutator identities involve signatures depending on the degrees of the elements. In particular, $L_{\bar{0}}$ is a Lie algebra and $L_{\bar{1}}$ is an $L_{\bar{0}}$-module.

According to Kac the classification of simple Lie superalgebras is divided into two main parts. In the first part we give the classification of the so-called classical simple Lie superalgebras. A simple Lie superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ is called classical if the representation of $L_{\bar{0}}$ in $L_{\bar{1}}$ is completely reducible. Remarkably enough it is exactly this class of Lie superalgebras to which the author and his co-workers were led from the physical side [6 - 8]. We have shown [6] that a simple Lie superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ is classical if and only if the Lie algebra $L_{\bar{0}}$ is reductive. Now the reductive Lie algebras are those which are commonly used to describe internal symmetries of elementary particles. Hence the classical simple Lie superalgebras seemed to be a reasonable family of algebras to be classified.

In the present work we shall deal with the classical simple Lie superalgebras by means of the techniques which have been developed in [7, 8] These have the advantage of starting directly from the Jacobi identity. In particular, they make quite evident "why" the two main double sequences of classical simple Lie superalgebras (the special linear and the orthosymplectic Lie superalgebras) exist. Regrettably, this method is not powerful enough to enable the classification of all classical simple Lie superalgebras, however, the remaining cases can be settled with a rather small amount of technicalities.

In the second part we obtain the classification of the non-classical simple Lie superalgebras. Here the filtrations and Z - gradations of Lie superalgebras play an important role. As an intermediate step we have to classify a certain family of transitive Z - graded Lie superalgebras which arise from filtrations of simple Lie superalgebras. The reasoning in this part is essentially due to Kac [3]. However, we have added several details and at some places simplified his arguments. In particular, we have avoided to use his theory of contragredient Lie superalgebras. Certainly, this theory is interesting in itself, but in our opinion it is not necessary (perhaps not even advantageous) to apply it in the proof of the classification theorem.

Our proof of the classification theorem appears to be somewhat incoherent in that we have to discuss various special cases. The reader might suspect that this is due to the fact that we have combined two different approaches to the subject. However, a short look at Kac's proof shows that this is not the case. Thus it is still worth-while to seek for some organizing principle which, finally, might allow of a uniform proof of the theorem.

Let us now give a brief account of the contents of this work. Chapter 0 is preparatory; we introduce our main conventions in §1 and make some general remarks on graded algebraic structures in §2.

Chapter I is formal in character. We give the basic definitions in §1, discuss the enveloping algebra of a Lie superalgebra in §2, and describe the usual elementary constructions with representations of Lie superalgebras in §3. Not all the material which is covered in §3 is really necessary for the rest of this work, however, it might be useful to have these constructions collected at some place. In §4 we introduce the concepts of induced and produced representations of Lie superalgebras and generalize the Guillemin, Sternberg theorem. This theorem will be important for the discussion of the non-classical simple Lie superalgebras.

Chapter II is devoted to the discussion of the simple Lie superalgebras and to the proof of the classification theorem. §1 is introductory; it provides some information on Z - gradations and filtrations of Lie superalgebras. In §2 we derive a few elementary properties of simple Lie superalgebras and prove the characterization of the classical simple Lie superalgebras which has been mentioned above. §3 contains several results on Lie superalgebras whose Killing form is non-degenerate. The next two paragraphs are devoted to the classical simple Lie superalgebras; they are described in §4 and classified in §5. The latter paragraph also contains some partial results on Z - graded Lie superalgebras. In §6 we give a detailed discussion of the so-called Cartan Lie superalgebras. (Let us remark that Kac has used the language of differential forms for describing these algebras.) §7 is devoted to the proof of one further partial result on Z - graded Lie superalgebras. In §8, at last, we are ready to classify the Z - graded Lie superalgebras which arise from certain filtrations of simple Lie superalgebras. The classification of the simple Lie superalgebras themselves is an easy consequence of the earlier results of this chapter.

In chapter III we give (without proofs) a survey of various further developments. §1 contains some results on superderivations of Clifford algebras and of Lie superalgebras, in §2 we make a few remarks on nilpotent, solvable and semi-simple Lie superalgebras, finally, in §3 we comment on the finite-dimensional representations of simple Lie superalgebras.

In general we shall presuppose a working knowledge of the theory of Lie algebras. A standard reference is Bourbaki's treatise [9 - 12]; our notational conventions as well as some special results are included in the appendix.

In the present work we shall not comment on the theory of supermanifolds or of Lie supergroups. The reader who is interested in these topics is referred to the literature [13 - 17].

# Chapter 0 Preparatory remarks

## § 1 Conventions

1) In the present work we are dealing exclusively with vector spaces and algebras over a (commutative) field K of *characteristic zero*. All additional assumptions on the vector spaces and algebras (for example : finite-dimensional) or on the field K (for example : algebraically closed) will be mentioned explicitly; sometimes this will be done once and for all at the beginning of a paragraph or section.

2) An algebra over K is by definition a vector space (over K) equipped with some distributive (i.e. bilinear) multiplication. All additional assumptions on this multiplication (for example : associativity) will be mentioned explicitly.

3) The associative algebras appearing in this work will always contain a unit element.

A homomorphism of an associative algebra A into an associative algebra B is always assumed to map the unit element of A onto the unit element of B.

Every module M over an associative algebra A is assumed to be unitary (i.e. the multiplication by the unit element of A is the identity mapping of M).

4) Our notational conventions on Lie algebras are collected in the appendix.

## § 2 Some general remarks on graded algebraic structures

In this paragraph we recall the definitions of certain graded algebraic structures. For a more detailed discussion of such objects we refer the reader to the literature (for example, see [18, 19]).

Let $\Gamma$ be one of the rings $Z$ (ring of integers) or $Z_2 = Z/2Z$ (ring of integers modulo 2). We shall only consider gradations with values in $\Gamma$. The two elements of $Z_2$ will be denoted by $\bar{0}$ (residue class of even integers) and $\bar{1}$ (residue class of odd integers). If $a \in Z$ then the integer $(-1)^a$ depends only on the residue class modulo 2 of $a$. Hence for all $\alpha \in Z_2$ the integer $(-1)^\alpha$ is well-defined.

1) Let $V$ be a vector space over the field $K$. A $\Gamma$-*gradation of the vector space* $V$ is a family $(V_\gamma)_{\gamma\in\Gamma}$ of subspaces of $V$ such that

$$V = \bigoplus_{\gamma\in\Gamma} V_\gamma \quad . \tag{2.1}$$

The *vector space* $V$ is said to be $\Gamma$-*graded* if it is equipped with a $\Gamma$-gradation.

An element of $V$ is called *homogeneous* of degree $\gamma$, $\gamma \in \Gamma$, if it is an element of $V_\gamma$. In the case $\Gamma = Z_2$ the elements of $V_{\bar{0}}$ (resp. $V_{\bar{1}}$) are also called *even* (resp. *odd*).

Every element $y \in V$ has a unique decomposition of the form

$$y = \sum_{\gamma\in\Gamma} y_\gamma \quad ; \quad y_\gamma \in V_\gamma \quad , \quad \gamma \in \Gamma \tag{2.2}$$

(where, of course, only finitely many $y_\gamma$ are different from zero). The element $y_\gamma$ is called the *homogeneous component* of $y$ of degree $\gamma$.

A subspace $U$ of $V$ is called $\Gamma$-graded (or simply graded) if it contains the homogeneous components of all of its elements, i.e. if

$$U = \bigoplus_{\gamma\in\Gamma} (U \cap V_\gamma) \quad . \tag{2.3}$$

On any $Z$-graded vector space $V = \bigoplus_{j\in Z} V_j$ there exists a natural $Z_2$-gradation which is said to be induced by the $Z$-gradation and which is defined by

$$V_{\bar{0}} = \bigoplus_{j\in Z} V_{2j} \quad ; \quad V_{\bar{1}} = \bigoplus_{j\in Z} V_{2j+1} \quad . \tag{2.4}$$

2) Let

$$W = \bigoplus_{\gamma\in\Gamma} W_\gamma \tag{2.5}$$

be a second $\Gamma$-graded vector space. A linear mapping

$$g : V \longrightarrow W \tag{2.6}$$

is said to be *homogeneous* of degree $\gamma$, $\gamma \in \Gamma$, if

$$g(V_\alpha) \subset W_{\alpha+\gamma} \quad \text{for all } \alpha \in \Gamma \quad . \tag{2.7}$$

The mapping $g$ is called a homomorphism of the $\Gamma$-graded vector space $V$ into the $\Gamma$-graded vector space $W$ if $g$ is homogeneous of degree $0$. It is now evident how we define an isomorphism or an automorphism of $\Gamma$-graded vector spaces.

3) Let $U$ and $U'$ be two $\Gamma$-graded vector spaces over $K$. Then the tensor product $U \otimes U'$ has a natural $\Gamma$-gradation which is defined by

$$(U \otimes U')_\gamma = \bigoplus_{\alpha+\beta=\gamma} (U_\alpha \otimes U'_\beta) \quad , \quad \gamma \in \Gamma \quad . \tag{2.8}$$

Suppose that we are given in addition two $\Gamma$-graded vector spaces $V$ and $V'$. Let $g : U \longrightarrow V$ and $g' : U' \longrightarrow V'$ be two linear mappings which are homogeneous of degrees $\gamma$ and $\gamma'$, respectively. We define a linear mapping

$$g \mathbin{\bar{\otimes}} g' : U \otimes U' \longrightarrow V \otimes V' \tag{2.9,a}$$

by the requirement that

$$(g \mathbin{\bar{\otimes}} g')(x \otimes x') = (-1)^{\gamma'\xi}\, g(x) \otimes g'(x') \tag{2.9,b}$$

$$\text{for all } x \in U_\xi \, , \; x' \in U' \; ; \; \xi \in \Gamma \; .$$

Evidently, $g \mathbin{\bar{\otimes}} g'$ is a homogeneous linear mapping of degree $\gamma + \gamma'$.

If we are given two more $\Gamma$-graded vector spaces $W$ and $W'$ as well as two linear mappings $h : V \longrightarrow W$ and $h' : V' \longrightarrow W'$ which are homogeneous of degrees $\delta$ and $\delta'$, respectively, then we have

$$(h \,\bar{\otimes}\, h') \circ (g \,\bar{\otimes}\, g') = (-1)^{\delta'\gamma} (h \circ g) \,\bar{\otimes}\, (h' \circ g') \quad . \tag{2.10}$$

Remark 1)

Let $\langle \, , \, \rangle$ denote the "super-commutator" as defined in chapter I, §1, example 2). Suppose that $U = V = W$ and $U' = V' = W'$; if $\langle h , g \rangle = 0$ and $\langle h' , g' \rangle = 0$ it follows that $\langle h \,\bar{\otimes}\, h' , g \,\bar{\otimes}\, g' \rangle = 0$, too.

It is obvious how to generalize all these results to tensor products of finitely many $\Gamma$-graded vector spaces.

4) Let $A$ be an algebra over $K$. The *algebra* $A$ is said to be $\Gamma$-*graded* if the underlying vector space of $A$ is $\Gamma$-graded,

$$A = \bigoplus_{\gamma \in \Gamma} A_\gamma \quad , \tag{2.11}$$

and if, furthermore,

$$A_\alpha A_\beta \subset A_{\alpha+\beta} \quad \text{for all } \alpha , \beta \in \Gamma \quad . \tag{2.12}$$

Evidently, $A_0$ is a subalgebra of $A$. If $A$ has a unit element then this element lies in $A_0$.

A homomorphism of $\Gamma$-graded algebras is by definition a homomorphism of the underlying algebras as well as of the underlying $\Gamma$-graded vector spaces; in particular, a homomorphism is homogeneous of degree $0$. Similar remarks apply for isomorphisms and automorphisms.

A graded subalgebra (resp. ideal) of a $\Gamma$-graded algebra $A$ is a subalgebra (resp. ideal) of the algebra $A$ which is, in addition, a graded subspace of the $\Gamma$-graded vector space $A$. The quotient algebra of a $\Gamma$-graded algebra modulo a (two-sided) graded ideal is again a $\Gamma$-graded algebra.

If $A$ and $B$ are two $\Gamma$-graded algebras the direct (i.e. the cartesian) product $A \times B$ is an algebra which becomes a $\Gamma$-graded algebra by means of the definition

$$(A \times B)_\gamma = A_\gamma \times B_\gamma \quad \text{for all } \gamma \in \Gamma \ . \tag{2.13}$$

5) Let A be a Z-graded algebra and let A' be the Z-graded algebra whose underlying algebra is equal to that of A but whose Z-gradation is given by

$$A'_j = A_{-j} \quad \text{for all } j \in Z \ . \tag{2.14}$$

Then A' is called the Z-graded algebra obtained from A by inversion of the Z-gradation. Note that according to our definitions the Z-graded algebras A and A' are not necessarily isomorphic.

6) Let A and B be two $\Gamma$-graded associative algebras (recall that according to our conventions this implies that A and B have a unit element). On the $\Gamma$-graded vector space $A \otimes B$ (see 3)) we define a multiplication by the requirement that

$$(a \otimes b)(a' \otimes b') = (-1)^{\beta\alpha'}(a a') \otimes (b b') \tag{2.15}$$

$$\text{for all } a \in A \, , \ b \in B_\beta \, , \ a' \in A_{\alpha'} \, , \ b' \in B \ ; \ \beta \, , \alpha' \in \Gamma \ .$$

It is easy to check that with this multiplication $A \otimes B$ is a $\Gamma$-graded associative algebra [19]. This algebra will be called the *graded tensor product* of the $\Gamma$-graded algebras A and B and will be denoted by $A \bar{\otimes} B$ (in order to avoid a confusion with the more usual definition).

The algebras $A \bar{\otimes} B$ and $B \bar{\otimes} A$ are canonically isomorphic. In fact, it is easy to see that there exists a unique linear mapping

$$s : A \bar{\otimes} B \longrightarrow B \bar{\otimes} A \tag{2.16,a}$$

such that

$$s(a \otimes b) = (-1)^{\alpha\beta}\, b \otimes a \tag{2.16,b}$$

$$\text{for all } a \in A_\alpha \, , \ b \in B_\beta \ ; \ \alpha \, , \beta \in \Gamma$$

and that this mapping is an isomorphism of $\Gamma$-graded algebras.

The definition of the graded tensor product of associative $\Gamma$-graded algebras is easily generalized to the case of more than two factors [19];

this construction is still "associative" in the usual sense.

7) Let A be a $\Gamma$ - graded associative algebra and let V be a left A - module (recall that according to our conventions A has a unit element and that V is unitary). In particular, V is a vector space over K . The A - *module* V is said to be $\Gamma$ - *graded* if the underlying vector space of V is $\Gamma$ - graded and if, furthermore,

$$A_\alpha V_\beta \subset V_{\alpha+\beta} \quad \text{for all } \alpha, \beta \in \Gamma \quad . \tag{2.17}$$

Right $\Gamma$ - graded A - modules are defined similarly.

A homomorphism of $\Gamma$ - graded A - modules is by definition a homomorphism of the underlying A - modules as well as of the $\Gamma$ - graded vector spaces (i.e. it is A - linear and homogeneous of degree 0 ). Similar remarks apply for isomorphisms and automorphisms.

8) Let A and B be two $\Gamma$ - graded associative algebras and let V (resp. W ) be a $\Gamma$ - graded left A - module (resp. B - module). Then $V \otimes W$ is a $\Gamma$ - graded vector space and $A \bar{\otimes} B$ is a $\Gamma$ - graded associative algebra (see 3) and 6)). It is easy to see that there exists a unique structure of a left $\Gamma$ - graded $A \bar{\otimes} B$ - module on $V \otimes W$ such that

$$(a \otimes b)(x \otimes y) = (-1)^{\beta\xi} (a x) \otimes (b y) \tag{2.18}$$

for all $a \in A$ , $b \in B_\beta$ , $x \in V_\xi$ , $y \in W$ ; $\beta, \xi \in \Gamma$ .

9) Finally, we shall introduce some notions for the case where an algebra is equipped with both a $Z_2$ - gradation and a Z - gradation. It is convenient to define [ 3 ] :

Definition 1

A $Z_2$ - graded algebra is called a *superalgebra.*

Definition 2

A *superalgebra* S is said to be Z - *graded* if we are given a family $(S_j)_{j\in Z}$ of $Z_2$ - graded subspaces of S such that

$$S = \bigoplus_{j \in Z} S_j \qquad (2.19,a)$$

$$S_i S_j \subset S_{i+j} \quad \text{for all } i\,,\, j \in Z \quad . \qquad (2.19,b)$$

The Z - gradation $(S_j)_{j \in Z}$ is said to be *consistent* with the $Z_2$ - gradation of S if

$$S_{\bar{0}} = \bigoplus_{j \in Z} S_{2j} \quad ; \quad S_{\bar{1}} = \bigoplus_{j \in Z} S_{2j+1} \quad . \qquad (2.20)$$

According to this definition a Z - graded superalgebra is just a $\Delta$ - graded algebra, $\Delta$ being the additive group $Z \times Z_2$ . Furthermore, a superalgebra with a consistent Z - gradation (or, as we shall say, a consistently Z - graded superalgebra) is nothing but a Z - graded algebra which is equipped in addition with the $Z_2$ - gradation induced by its Z - gradation.

# Chapter I Formal constructions

## § 1 Definition and elementary properties of Lie superalgebras

Recall (chapter 0, definition 1) that a superalgebra is by definition nothing else but a $Z_2$-graded algebra.

Definition 1

Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a superalgebra whose multiplication is denoted by a pointed bracket $\langle \ , \ \rangle$. This implies in particular that

$$\langle L_\alpha , L_\beta \rangle \subset L_{\alpha+\beta} \quad \text{for all } \alpha , \beta \in Z_2 \ . \tag{1.1}$$

We call L a *Lie superalgebra* if the multiplication satisfies the following identities:

$$\langle A , B \rangle = -(-1)^{\alpha\beta} \langle B , A \rangle \tag{1.2}$$

(graded skew-symmetry)

$$(-1)^{\gamma\alpha}\langle A , \langle B , C \rangle\rangle + (-1)^{\alpha\beta}\langle B , \langle C , A \rangle\rangle + (-1)^{\beta\gamma}\langle C , \langle A , B \rangle\rangle = 0$$

(graded Jacobi identity) (1.3)

for all $A \in L_\alpha$, $B \in L_\beta$, $C \in L_\gamma$ ; $\alpha , \beta , \gamma \in Z_2$ .

Remark 1)

Lie superalgebras are frequently called $Z_2$-graded Lie algebras in spite of the fact that in general they are *not* Lie algebras.

Let L be a Lie superalgebra. All the following statements are obvious.

a) The subalgebra $L_{\bar{0}}$ of L is a Lie algebra.

b) If the graded vector space L is equipped with the "inverted multiplication" $(A , B) \longrightarrow \langle B , A \rangle$ we obtain again a Lie superalgebra.

c) The definitions of a graded subalgebra, a graded ideal, a graded quotient algebra of L are standard and need not be repeated here (see chap-