Level Crossing Methods in Stochastic Models

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"Out of great complexity
comes great simplicity."

adapted from Winston Churchill

To the memory of my parents

PREFACE

From 1972 to 1974, I was working on a PhD thesis entitled *Multiple Server Queues with Service Time Depending on Waiting Time*. The method of analysis was the embedded Markov chain technique, described in the papers [82] and [77]. My analysis involved lengthy, tedious derivations of systems of integral equations for the probability density function (pdf) of the waiting time. After pondering for many months whether there might be a faster, easier way to derive the integral equations, I finally discovered the basic theorems for such a method in August, 1974. The theorems establish a connection between sample-path level-crossing rates of the virtual wait process and the pdf of the waiting time. This connection was not found anywhere else in the literature at the time. I immediately developed a comprehensive new methodology for deriving the integral equations based on these theorems, and called it *system point theory.* (Subsequently it was called *system point method*, or *system point level crossing method: SPLC* or simply *LC*.) I rewrote the entire PhD thesis from November 1974 to March 1975, using LC to reach solutions. The new thesis was called *System Point Theory in Exponential Queues.* On June 12, 1975 I presented an invited talk on the new methodology at the Fifth Conference on Stochastic Processes and their Applications at the University of Maryland. Many queueing theorists were present. Ever since, LC has become an increasingly used technique for analyzing a large class of stochastic models. LC can be used to derive integro-differential equations for transient distributions, or integral equations for steady-state distributions.

This monograph elucidates LC for obtaining probability distributions of state variables in a variety of stochastic models. Most of the analyses are for steady-state distributions. However, some results for transient distributions are also given. The book is intended for research- and applications-oriented workers in operations research, management science, engineering, probability and statistics, actuarial science, mathematics, and the natural sciences.

To date, many researchers have applied LC. Applications have appeared in refereed journals, conference Proceedings, technical reports, Masters and PhD theses, and in chapters and sections of books, worldwide.

One reason for this great interest and consequent proliferation of publications, is that LC is very intuitive. Furthermore, it leads to exact analytical solutions. An LC analysis starts with a typical sample path of a stochastic process. A sample path (sample function, realization, tracing) can be thought of dynamically. That is, the path evolves in the state space over time, governed by the probability laws of the model.

The LC method focuses on *time rates* at which a sample path exits and enters certain measurable state-space sets. Level-crossing theorems equate these transition rates to simple algebraic expressions of the pdf and/or cdf (cumulative distribution function) of the state variable. In a steady-state analysis, the algebraic expressions often appear in separate terms of Volterra integral equations of the second kind with parameter. Thus, "physical" sample-path transition rates are in one-to-one correspondence with terms of the integral equations. The integral equations themselves are constructed by applying rate conservation laws, e.g., rate balance. The upshot is that we can write down the integral equations "by inspection", upon observing the sample-path structure of a model!

The integral equations are solved simultaneously with a normalizing condition, which specifies that all probabilities sum to 1. The system of equations is solved for the pdf and/or cdf of the state variable. We may use analytical, numerical, algorithmic, simulation, or approximation techniques to solve the system of equations. We can derive operating characteristics of the model using the solution and/or LC concepts.

It is axiomatic that one can reach solutions for mathematical models by applying alternative techniques. My own experience, and that of many other researchers, has demonstrated that LC often leads quickly and easily to solutions. It provides useful intuition about the model dynamics. This is due to the perspective taken: geometric sample-path structure; rate conservation laws; connection to concepts of natural science such as Physics. LC may free the analyst from lengthy derivations of a system of model equations. Thus it facilitates focusing on model dynamics and on operating characteristics. An LC analysis quite often suggests new creative approaches for studying a model.

Chapter 1 outlines the original developmental ideas which led me to the discovery of LC. When combined synergistically, the basic ideas lead to a powerful modelling technique.

Chapter 2 defines and discusses basic concepts relevant to the method, such as: state space, sample path, system point (SP), SP jump, state-space level, boundary, downcrossing, upcrossing, tangent, etc.

Chapters 3, 4, and 5 analyze steady-state distributions in variants of M/G/1, M/M/c and G/M/c queues, respectively. Chapters 3 and 4 also provide some basic results for transient distributions.

Chapter 6 analyzes steady-state distributions in several basic dams, and in two inventory models. It also includes some transient results.

Chapter 7 demonstrates a multi-dimensional technique with applications to two 2-dimensional inventory models.

Chapter 8 explains the embedded level crossing technique with applications to dams and queues.

Chapter 9 gives an introduction to level crossing estimation, which uses simulation of sample paths to obtain solutions.

Chapter 10 applies LC to a variety of models including: a replacement model, renewal theory, Markov renewal theory, Markov chains, growth and counter models, a dam with alternating continuous influx and efflux, simple harmonic motion. It also illustrates some transient analyses.

I hope that readers will find the monograph interesting, and useful for research. The concepts, techniques, examples, applications and theoretical results in this book may suggest potentially new theory and new applications.

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The late Dr. Julian Keilson was the external examiner for my PhD thesis. He independently initiated a formal written invitation from the conference chair Dr. R. Syski, for me to make the first international conference presentation on the level crossing methodology at the Fifth Conference on Stochastic Processes and their Applications in June, 1975.

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