THE SCHUR COMPLEMENT AND ITS APPLICATIONS

Numerical Methods and Algorithms

VOLUME 4

Series Editor:

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THE SCHUR COMPLEMENT AND ITS APPLICATIONS

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Springer

Library of Congress Cataloging-in-Publication Data

A C.I.P. record for this book is available from the Library of Congress.

ISBN 0-387-24271-6 e-ISBN 0-387-24273-2 Printed on acid-free paper.

Printed in the United States of America.

9 8 7 6 5 4 3 2 1 SPIN 11161356

springeronline.com

To our families, friends, and the matrix community

Issai Schur (1875-1941)

This portrait of Issai Schur was apparently made by the "Atelieir Hanni Schwarz, N. W. Dorotheenstraße 73" in Berlin, *c.* 1917, and appears in *Ausgewählte Arbeiten zu den Ursprüngen der Schur-Analysis: Gewidmet dem großen Mathematiker Issai Schur (1875–1941)* edited by Bernd Fritzsche & Bernd Kirstein, pub. B. G. Teubner Verlagsgesellschaft, Stuttgart, 1991.

Emilie Virginia Haynsworth (1916-1985)

This portrait of Emilie Virginia Haynsworth is on the Auburn University Web site www.auburn.edu/~fitzpjd/ben/images/Emilie.gif and in the book *The Education of a Mathematician* by Philip J. Davis, pub. A K Peters, Natick, Mass., 2000.

Contents

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Preface

What's in a name? To paraphrase Shakespeare's Juliet, that which Emilie Haynsworth called the *Schur complement*, by any other name would be just as beautiful. Nevertheless, her 1968 naming decision in honor of Issai Schur (1875–1941) has gained lasting acceptance by the mathematical community. The Schur complement plays an important role in matrix analysis, statistics, numerical analysis, and many other areas of mathematics and its applications.

Our goal is to expose the Schur complement as a rich and basic tool in mathematical research and applications and to discuss many significant results that illustrate its power and fertility. Although our book was originally conceived as a research reference, it will also be useful for graduate and upper division undergraduate courses in mathematics, applied mathematics, and statistics. The contributing authors have developed an exposition that makes the material accessible to readers with a sound foundation in linear algebra.

The eight chapters of the book (Chapters 0–7) cover themes and variations on the Schur complement, including its historical development, basic properties, eigenvalue and singular value inequalities, matrix inequalities in both finite and infinite dimensional settings, closure properties, and applications in statistics, probability, and numerical analysis. The chapters need not be read in the order presented, and the reader should feel at leisure to browse freely through topics of interest.

It was a great pleasure for me, as editor, to work with a wonderful group of distinguished mathematicians who agreed to become chapter contributors: T. Ando (Hokkaido University, Japan), C. Brezinski (Université des Sciences et Technologies de Lille, France), R. A. Horn (University of Utah, Salt Lake City, USA), C. R. Johnson (College of William and Mary, Williamsburg, USA), J.-Z. Liu (Xiangtang University, China), S. Puntanen (University of Tampere, Finland), R. L. Smith (University of Tennessee, Chattanooga, USA), and G. P. H. Styan (McGill University, Canada).

I am particularly thankful to George Styan for his great enthusiasm in compiling the master bibliography for the book. We would also like to acknowledge the help we received from Gülhan Alpargu, Masoud Asgharian, M. I. Beg, Adi Ben-Israel, Abraham Berman, Torsten Bernhardt, Eva Brune, John S. Chipman, Ka Lok Chu, R. William Farebrother, Bernd Fritsche, Daniel Hershkowitz, Jarkko Isotalo, Bernd Kirstein, André Klein, Jarmo Niemelä, Geva Maimon Reid, Timo Mäkeläinen, Lindsey E. McQuade, Aliza K. Miller, Ingram Olkin, Emily E. Rochette, Vera Rosta, Eugénie Roudaia, Burkhard Schaffrin, Hans Schneider, Shayle R. Searle, Daniel N. Selan, Samara F. Strauber, Evelyn M. Styan, J. C. Szamosi, Garry J. Tee, Götz Trenkler, Frank Uhlig, and Jürgen Weiß. We are also very grateful to the librarians in the McGill University Interlibrary Loan and Document Delivery Department for their help in obtaining the source materials for many of our references. The research of George P. H. Styan was supported in part by the Natural Sciences and Engineering Research Council of Canada.

Finally, I thank my wife Cheng, my children Sunny, Andrew, and Alan, and my mother-in-law Yun-Jiao for their understanding, support, and love.

Fuzhen Zhang
September 1, 2004
Fort Lauderdale, Florida

Chapter 0

Historical Introduction: Issai Schur and the Early Development of the Schur Complement

0.0 Introduction and mise-en-scène

In this introductory chapter we comment on the history of the Schur complement from 1812 through 1968 when it was so named and given a notation. As Chandler & Magnus [113, p. 192] point out, "The coining of new technical terms is an absolute necessity for the evolution of mathematics." And so we begin in 1968 when the mathematician Emilie Virginia Haynsworth (1916–1985) introduced a name and a notation for the Schur complement of a square nonsingular (or invertible) submatrix in a partitioned (two-way block) matrix [210, 211].

We then go back fifty-one years and examine the seminal lemma by the famous mathematician Issai Schur (1875–1941) published in 1917 [404, pp. 215–216], in which the *Schur determinant formula* (0.3.2) was introduced. We also comment on earlier implicit manifestations of the Schur complement due to Pierre Simon Laplace, later Marquis de Laplace (1749–1827), first published in 1812, and to James Joseph Sylvester (1814–1897), first published in 1851.

Following some biographical remarks about Issai Schur, we present the *Banachiewicz inversion formula* for the inverse of a nonsingular partitioned matrix which was introduced in 1937 [29] by the astronomer Tadeusz Banachiewicz (1882–1954). We note, however, that closely related results were obtained earlier in 1933 by Ralf Lohan [290], following results in the book [66] published in 1923 by the geodesist Hans Boltz (1883–1947).

We continue with comments on material in the book *Elementary Matrices and Some Applications to Dynamics and Differential Equations* [171], a