Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

Subseries Institut de Mathématiques, Université de Strasbourg
Adviser P.A. Meyer

1059

Séminaire de Probabilités XVIII 1982/83

Proceedings

Edité par J. Azéma et M. Yor

Springer-Verlag
Berlin Heidelberg New York Tokyo 1984

Editeurs

Jacques Azéma
Marc Yor
Laboratoire de Probabilités
4 Place Jussieu, Tour 56, 75230 Paris Cédex 05 – France

AMS Subject Classification (1980): 60GXX, 60HXX, 60JXX

ISBN 3-540-13332-1 Springer-Verlag Berlin Heidelberg New York Tokyo
ISBN 0-387-13332-1 Springer-Verlag New York Heidelberg Berlin Tokyo

Printed in Germany

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.
2146/3140-543210

SEMINAIRE DE PROBABILITES XVIII

TABLE DES MATIERES

F.S. De Blasi, Istituto Matematico Università, Viale Morgagni 67/A, 50134 Firenze

F. Degan, Istituto di Matematica Applicata, Via Belzoni 7, 35131 Padova

A. De Vito, Via Trieste 19, 60124 Ancona

B. D'Onofrio, Via V. Veneto 7, 04020 SS. Cosma e Damiano (Latina)

H. Duistermaat, Mathematical Institute, University of Utrecht,
Budapestlaan 6, Utrecht (NL)

K. Dysthe, I.M.R. University of Tromso, 9001 Tromso, Norway

M. Falcone, CEREMADE, University Paris IX Dauphine, 75775 Paris Cedex, France

A. Fasano, Istituto Matematico Università, Viale Morgagni 67/A, 50134 Firenze

C. Fortier, 2705 Blv. Liégeois, Ste-Foy, Québèc, Canada

L. Fronzoni, Istituto di Fisica, Università, 56100 Pisa

M. Furi, Istituto di Matematica Applicata, Facoltà di Ingegneria, Via S. Marta 3,
50139 Firenze

G. Fusco, Istituto di Matematica Applicata, Università, Via A. Scarpa 10, 00161 Roma

M. Galeotti, Istituto di Matematica Applicata, Facoltà di Ingegneria,
Via S. Marta 3, 50139 Firenze

L. Galgani, Dipartimento di Matematica, Università, Via Saldini 50, 20133 Milano

J.J. Gervais, Départment de Mathématiques, Université Laval, Québèc, Canada

S. van Gils, Mathematical Centre, P.O. Box 4079, 1009 AB Amsterdam, Holland

A. Giorgilli, Dipartimento di Fisica, Via Celoria 16, 20133 Milano

F. Gori, Via Cimabue 49, 50100 Firenze

J. de Graaf, Eindhoven University of Technology, Postbus 513,
5600 Eindhoven, The Netherlands

P. Hagedorn, Institut fur Mechanik, TH Darmstadt, D-6100 Darmstadt, Germany

J.K. Hale, Division of Applied Mathematics, Brown University,
Providence, R.I. 02912, USA

G. Iooss, Inst. de Math. et Sc. Phys., Université de Nice,
Parc Valrose, 06034 Nice, France

S. Larsson, Department of Mathematics, Chalmers University of Technology,
S-412 96 Goteborg, Sweden

W. Loud, Department of Mathematics, University of Minnesota, Minneapolis,
Minnesota 55455, USA

C. Maffei, Viale Angelico, 70, 00195 Roma

P. Marcati, Dipartimento di Matematica, Università di Trento, 38050 Povo, Trento

G. Marmo, Dipartimento di Matematica, Università di Trento, 38050 Povo, Trento

G.F. Mattioli, Istituto Matematico Università, P.le A. Moro 2, 00185 Roma

G.P. McCauley, Department of Mathematics, University of Birmingham, P.O.Box 363,
Birmingham B15 2TT, England

J. van der Meer, Mathematical Center, P.O.Box 4079, 1009 AB Amsterdam, Holland

M. Miari, Via Papa Giovanni 24, 20090 Segrate, Milano

M. Mininni, Viale Federico II 15, 70038 Terlizzi (Bari)

V. Moauro, Dipartimento di Matematica, Università di Trento, 38050 Povo, Trento

J. Myjak, Instytut Matematyki AGH, al. Mickyuwicz 30, 30059 Krakow, Poland

P. Negrini, Viale Angelico 70, 00195 Roma

P. Nistri, Istituto di Matematica Applicata, Facoltà di Ingegneria,
Via S. Marta 3, 50139 Firenze

E. Omerti, Dipartimento di Matematica, Università di Trento, 38050 Povo, Trento

L. Pandolfi, Dipartimento di Matematica, Politecnico di Torino
Corso Duca degli Abruzzi 24, 10129 Torino

R. Pavani, Istituto G. Donegani, Via Fauser 4, 28100 Novara

P. Pera, Istituto di Matematica Applicata, Facoltà di Ingegneria,
Via S. Marta 3, 50139 Firenze

R. Raugel, 6 Blv. Jourdan, 75014 Paris, France

R. Ricci, Istituto Matematico Università, Viale Morgagni 67/A, 50134 Firenze

R.M. Roberts, Institut des Hautes Etudes Scientifiques, 91440 Bures-sur-Yvette,

France

M. Sabatini, Viale XX Settembre 56, 34126 Trieste

L. Salvadori, Dipartimento di Matematica, Università di Trento, 38050 Povo, Trento

A. Salvatore, Via Calefati 266, 70100 Bari

J.A. Sanders, Wiskundig Seminarium VU, Postbus 7161, 1007 Amsterdam, Holland

M. Scalia, Via F. D'Ovidio 16/A, 00137 Roma

D. Socolescu, Mathystrasse 27, D-7500 Karlsruhe 1

G. Stefani, Istituto di Matematica Applicata, Facoltà di Ingegneria
Via S. Marta 3, 50136 Firenze

D. Stoffer, Seminar fur Angewandte Mathematik, ETH-Zentrum, CH-8092 Zurich

S. van Strien, Onderafdeling Wiskund, Technische Hogeschool Delft,
Delft, The Netherlands

C. Valente, Via S. Godenzo 36, 00100 Roma

A. Vanderbauwhede, Institute for Theoretical Mechanics, Krijgslaan 281,
B-9000 Gent, Belgium

J.M. Vegas, Departamento de Ecuaciones Funcionales, Universidad Complutense,
Madrid 3, Spain

V. Vespri, Via Pisacane 11, Viareggio (Lucca)

F. Visentin, Piazza Medaglie d'Oro 35, 80129 Napoli

P. Zecca, Dipartimento di Informatica e Sistemistica, Via S. Marta 3, 50139 Firenze

P. Zezza, Dipartimento di Informatica e Sistemistica, Via S. Marta 3, 50139 Firenze

F. Zirilli, Istituto Matematico Università, 84100 Salerno

TABLE OF CONTENTS

Levels at which every Brownian excursion is exceptional

M.T. Barlow, E. Perkins

Statistical Laboratory
16 Mill Lane
Cambridge
CB2 1SB
England

Dept. of Mathematics
University of British Columbia
Vancouver
British Columbia
V6T 1Y4
Canada

1. Introduction. Let B_t be a Brownian motion starting at 0, defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ which satisfies the usual conditions. For each $x \in \mathbb{R}$, let $\Lambda^x(B)$ be the set of starting times of excursions of B above the level x, and let $\Lambda(B)$ be the set of starting times of excursions of B above some level: that is

$\Lambda^x(B.(\omega)) = \Lambda^x(B) = \{t : B_t = x,\ B_s > x$ for $t < s < t + \varepsilon$ for some $\varepsilon > 0\}$,

$\Lambda = \Lambda(B) = \bigcup_x \Lambda^x(B)$.

Let $t \in \Lambda(B(\omega))$, and $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous strictly increasing function with $f(0) = 0$. We will say that f is a <u>lower function</u> (respectively, <u>upper function</u>) for B at t if there is a $\delta(\omega) > 0$ such that

$$B_{t+u}(\omega) - B_t(\omega) \geq f(u) \qquad \text{for all} \quad u \in [0,\delta]$$

(respectively,

$$B_{t+u}(\omega) - B_t(\omega) \leq f(u) \qquad \text{for all} \quad u \in [0,\delta]) \ .$$

If $A \subset \Lambda$, and f is a lower (upper) function for B at t for all $t \in A$, we will say that f is a uniform lower (upper) function for B on A .

For fixed x , it is well known which functions f are upper or lower functions on Λ^x . Let X be a 3-dimensional Bessel process (we will say a Bes (3) process). It follows from the decomposition of Brownian excursions given in [9, p. 74] that, with probability 1, each excursion of B from x begins in the same way that X leaves 0 . As there are only countably many excursions of B from x , a function f is , a.s., a uniform lower, or upper, function for B on Λ_x if and only if f is a lower, or upper, function for X at 0 . An integral test for this last property is known (see [11, p. 144, 147]). Let $f(t) = t^{1/2}\phi(t)$. Then f is an upper function for X at 0 if $\phi(t) \uparrow \infty$ as $t \downarrow 0$, and $\int_{0+} \phi^3(t)e^{-(1/2)\phi^2(t)}t^{-1}dt < \infty$, and f is a

lower function for X at 0 if $\phi(t) \downarrow 0$ as $t \downarrow 0$, and

$\int_{0+} \phi(t)t^{-1}dt < \infty$. In particular, for each $\varepsilon > 0$, $\sqrt{2}(1+\varepsilon)t^{1/2}(\log\log 1/t)^{1/2}$ is an upper function, and $t^{1/2}(\log 1/t)^{-(1+\varepsilon)}$ is a lower function.

By Fubini's theorem, if f is a lower function for X at 0 , then $\{x : f$ is a uniform lower function on $\Lambda^x\}$ is of full measure. However, there may be times $t \in \Lambda(\omega)$ for which f fails to be a lower function, and therefore levels x such that f fails to be a lower function for one or more excursions above x .

We may consider 4 types of 'bad behaviour':

(i) times at which functions f which are lower functions for X at 0 fail to be lower functions,

(ii) times at which functions f which are upper functions for X at 0 fail to be upper functions,

(iii) times at which functions f which are not lower functions for X at 0 are lower functions,

(iv) times at which functions f which are not upper functions for X at 0 are upper functions.

(i) In Section 2 we show that, given any continuous strictly increasing function f there are, a.s., times $t \in \Lambda(\omega)$ such that f fails to be a lower function at t . Remarkably, even more is true: there are levels x at which f fails to be a lower function for every excursion above that level x . In fact, this is a real-variable result, which is a consequence of Baire's category theorem: the only properties of Brownian motion that are used are that it is continuous and nowhere monotonic.

It is also of interest to consider the size of the sets on which some function fails to be a lower function. For $1/2 \leq p < \infty$ let

$$\Lambda_p = \{t \in \Lambda : \text{there exists } \delta_n \downarrow 0 \text{ with } B_{t+\delta_n} - B_t < \delta_n^p \text{ for all } n\} .$$

In Section 3 we show that $\dim \Lambda_p = 1/(4p)$. (Here dim denotes the Hausdorff dimension).

(ii) The Lévy modulus of continuity for Brownian motion provides a uniform upper function for B on Λ , but this is not quite the best possible. Let $\phi(t) = t^{1/2}(\log 1/t)^{1/2}$ (the Lévy modulus is $\sqrt{2}\,\phi$) . Then (Theorem 4.3), $(1+\varepsilon)\phi$ is a uniform upper function on Λ , and $(1-\varepsilon)\phi$ fails to be a uniform upper function. We also find the dimension of the set on which $\alpha\phi$, for $0 < \alpha < 1$, fails to be an upper function: we only state the result here (Theorem 4.4), as the proof is very similar to the proofs in Section 3.

(iii) and (iv). We shall not consider these here, as fairly precise results about this type of behaviour have recently been obtained elsewhere (see [2], [3], [15], [17]). The situation concerning (iii) is as follows: w.p. 1 there exist a dense set of times t such that $\liminf_{h\downarrow 0} (B(t+h) - B(t))h^{-1/2} = 1$ (see [3]) but there is no t for which the above lim inf is greater than one ([2]); in fact there is no t for which $B(t+h) - B(t) \geq \sqrt{h}$ for all $h \in [0,\Delta]$ for some $\Delta > 0$ ([3]) . Regarding (iv), it is shown in [17] that

$$\inf_{t\in\Lambda} \limsup_{h \to 0+} \frac{(B_{t+h}-B_t)}{\sqrt{h}} = c$$

where $c(> 1)$ is the smallest positive zero of the unique (up to a multiplicative constant) solution of

$$\frac{1}{2}\left(\frac{d^2}{dx^2} - x\frac{d}{dx}\right)\psi(x) = -\psi(x) \ , \ \psi \in L^2([0,\infty), e^{-x^2/2}\,dx)$$

$$\psi(0) = 0 \ .$$

2. Lower Functions

The key to the results of this section is a real variable theorem, (Proposition 2.1) which is an easy consequence of Baire's category theorem and may well be known.

If $g : \mathbb{R}_+ \to \mathbb{R}$ is a cadlag function, set $\Delta g(t) = g(t) - g(t-)$.

Proposition 2.1. *Let* $g_1, g_2, \ldots$ *be a sequence of cadlag functions with the property that, for each* i , $\{\Delta g_i > 0\}$ *is dense in* $[0, 1]$. *Let* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *be continuous, increasing with* $\phi(0) = 0$. *Then there is a set* $A \subset (0,1)$ *with the following properties*:

(i) *For all* $t \in A$ *and* $i \geq 1$ *there is a sequence* $t_{i,n} \downarrow t$ *with* $g_i(t_{i,n}) - g_i(t) > \phi(t_{i,n} - t)$ *for all* n .

(ii) A *is the countable intersection of open dense sets in* $(0,1)$.

(iii) A *is of second category in* $\mathbb{R}$ *(and in particular uncountable) and dense in* $[0,1]$.

Proof. Let $C_{i,n} = \{t \in [0,1]:$ for some $h \in (0,n^{-1})$ and some $\varepsilon > 0$,

$$g_i(t'+h) - g_i(t') > \phi(h) \text{ for all } t' \in (t-\varepsilon, t+\varepsilon)\}.$$

It is clear from the definition that $C_{i,n}$ is open. Let (a,b) be any interval in $[0,1]$: then there exists $s \in (a,b)$ with $\Delta g_i(s) > 0$. Choose $h < n^{-1}$ such that $0 < \phi(h) < \Delta g_i(s)$: then, as g_i has left limits, for some $\varepsilon > 0$ we have $g_i(u+h) - g_i(u) > \phi(h)$ for $s - \varepsilon < u < s$. Thus $C_{i,n} \cap (a,b)$ is non-empty, and so $C_{i,n}$ is dense in $[0,1]$.

Now let $A = \bigcap_{i,n} C_{i,n}$: the set A is the intersection of a countable number of open dense sets, and therefore, by Baire's theorem (see [4, p. 249]), is dense in $[0,1]$, and of the second category. If $t \in A$, the existence of $t_{i,n} \downarrow t$ with the desired properties is immediate from the definition of A . □

We say that a function $g : \mathbb{R}_+ \to \mathbb{R}$ is *nowhere monotone* if g is not monotone in any interval: that is, given $a < b$ there exist

$a < s_1 < s_2 < s_3 < b$ such that $g(s_2) > g(s_1) \vee g(s_3)$.

Theorem 2.2. *Let* $b : \mathbb{R}_+ \to \mathbb{R}$ *be continuous and nowhere monotone. Let* $f : \mathbb{R}_+ \to \mathbb{R}$ *be continuous, strictly increasing, and with* $f(0) = 0$. *There exists an uncountable dense set* S , *of the second category in* $\mathbb{R}$, *such that, for all* $x \in S$, *and* $t \in \Lambda^x(b)$, *there exist* $t_n \downarrow t$ *with*

$$b(t_n) - b(t) < f(t_n - t) .$$

Proof. For $0 \le r < s$ let

$$I^{r,s} = (\inf_{r \le u \le s} b(u) , b(s)) ,$$

$$g^{r,s}(x) = \sup\{u \in [r,s) : b(u) = x\} \quad \text{for} \quad x \in I^{r,s} .$$

As $b(g^{r,s}(x)) = x < b(s)$ for $x \in I^{r,s}$, the function $g^{r,s}$ is right continuous and increasing on $I^{r,s}$. Further, the set $\{\Delta g^{r,s} > 0\}$ is dense in $I^{r,s}$: if $g^{r,s}$ were continuous on an interval $[u,v]$, this would imply that b was monotone on $[g^{r,s}(u), g^{r,s}(v)]$.

Let ϕ be the inverse function to f ,

$$A^{r,s} = \{x \in I^{r,s} : \text{there exist } x_n \downarrow x \text{ with } g^{r,s}(x_n) - g^{r,s}(x) > \phi(x_n - x) \text{ for each } n\} ,$$

and $C^{r,s} = A^{r,s} \cup (c\ell(I^{r,s}))^c$.

By (ii) of Proposition 2.1 $C^{r,s}$ contains a countable intersection of open sets each dense in $\mathbb{R}$. Therefore the same is true of $S = \bigcap_{\substack{0 \le r < s \\ r,s \in \mathbb{Q}}} C^{r,s}$

By Baire's theorem S is an uncountable dense set of the second category in

$\mathbb{R}$. Let $x \in S$ and $t \in \Lambda^x(b)$. If $t' = \inf\{u > t : b(u) = x\}$, then $t' > t$ and there are rationals r, s with $0 \le r \le t < s < t'$. Since $x \in C^{r,s} \cap c\ell(I^{r,s})$, there is a sequence $x_n \downarrow x$ such that $g^{r,s}(x_n) - g^{r,s}(x) > \phi(x_n - x)$ for each n . Let $t_n = g^{r,s}(x_n)$: we have $t_n - t > \phi(x_n - x) = \phi(b(t_n) - b(t))$. Thus $b(t_n) - b(t) < f(t_n - t)$ for each n , and S has the required properties. □

As, with probability 1, $B(\omega)$ is nowhere monotone, we deduce immediately

Theorem 2.3. *Let* $f : \mathbb{R}_+ \to \mathbb{R}$ *be a continuous strictly increasing function with* $f(0) = 0$. *Then for* a.a. ω *there is an uncountable dense set* $S(\omega) \subseteq \mathbb{R}$ *such that for all* $x \in S(\omega)$, *and* $t \in \Lambda^x(B(\omega))$, *there exists a sequence* $t_n \downarrow t$ with

$$B_{t_n}(\omega) - B_t(\omega) < f(t_n - t) \quad \text{for all} \quad n .$$

We may also look at the starting times of all excursions of B from the level x . Let $\tilde{\Lambda}^x = \{t: B_t = x , B_s \neq x$ for $t < s < t + \varepsilon$ for some $\varepsilon > 0\}$. Then, by making a few obvious changes in the previous arguments, one obtains

Corollary 2.4. *Let* $f : \mathbb{R}_+ \to \mathbb{R}$ *be a continuous strictly increasing function with* $f(0) = 0$. *Then for* a.a. ω *there is an uncountable dense set* $S(\omega) \subseteq \mathbb{R}$ *such that for all* $x \in S(\omega)$, *and* $t \in \tilde{\Lambda}^x(B(\omega))$, *there exists a sequence* $t_n \downarrow t$ *with*

$$|B_{t_n}(\omega) - B_t(\omega)| < f(t_n - t) \quad \text{for all} \quad n .$$

Remark 2.5. It is clear that these last two results also hold for any process with sample paths which are continuous and nowhere monotone. There are continuous Gaussian processes for which the local and global modulus of continuity are identical - see Kahane [10]: nevertheless they still exhibit this kind of sample function irregularity.

Let $f(t) = \exp(-1/t^2)$, and $S(\omega)$ be the set obtained in Corollary 2.4. For $x \in S(\omega)$, we see that every excursion from x begins in an unusually "slow" fashion, and this might suggest that there are asymptotically more excursions of small duration from x than at a typical level.

In fact, this is not the case. If $N_\varepsilon(t,x)$ is the number of excursions from x exceeding ε in length completed by B before time t , then in [13] it is shown that

$$(2.1) \qquad \lim_{\varepsilon \downarrow 0} \sup_{x \in \mathbb{R},\ t \le T} \left| \left(\tfrac{1}{2}\pi\varepsilon\right)^{1/2} N_\varepsilon(t,x) - L_t^x \right| = 0 \quad \text{for all } T \ge 0 \text{ , a.s.,}$$

where L_t^x is the local time of B at x . This extends a well known result of Lévy [12]. Other characterisations of Brownian local time that hold uniformly in x are given in [1] and [14].

In the light of these positive results it is of interest to note that Corollary 2.4 leads to a characterisation of local time that holds a.s. for any fixed level x , and therefore holds a.s. on a set of full Lebesgue measure, but which fails miserably on the uncountable dense set S .

Example 2.6. Recall $f(t) = \exp\{-1/t^2\}$. Let

$$C^x = \{\omega : \mathbb{R}_+ \to \mathbb{R} : \omega \text{ is continuous, } \omega(0) = x \text{ and there exist}$$
$$t_n \downarrow 0 \text{ with } |\omega(t_n) - x| < f(t_n) \text{ for each } n\} \text{ ,}$$

and let $N'_\varepsilon(t,x)$ be the number of excursions from x in $(C^x)^c$ that are of length greater than ε, and are completed by B by time t. As f is a lower function for the Bes(3) process, $N'_\varepsilon(t,x) = N_\varepsilon(t,x)$ for all $\varepsilon > 0$, $t \geq 0$ a.s., for each fixed x. Hence, by (2.1)

$$\lim_{\varepsilon \downarrow 0} (\tfrac{1}{2}\pi\varepsilon)^{1/2} N'_\varepsilon(t,x) = L^x_t \quad \text{for all } t \geq 0 \text{, a.s. for all } x \in \mathbb{R} .$$

However, by Corollary 2.4, for each $x \in S(\omega)$,

$$\lim_{\varepsilon \downarrow 0} (\tfrac{1}{2}\pi\varepsilon)^{1/2} N'_\varepsilon(t,x) = 0 .$$

Clearly, we could replace f in the definition of C^x above by any lower function for the Bes(3) process at 0.

3. Hausdorff Dimension and Lower Functions

Recall that $\Lambda_p = \{t \in A :$ there exist $t_n \downarrow t$ with $B_{t_n} - B_t < (t_n - t)^p$ for all $n\}$. In this section we find the Hausdorff dimension of the set Λ_p. Our main result (Theorem 3.6) could be proved using the "first method" of Orey and Taylor [16]. Indeed, our proof of the upper bound for the Hausdorff dimension follows their argument very closely. Their proof for the lower bound is more involved. We present a different argument here.

Lemma 3.1. Let $\tau(t)$ be a stable subordinator of index α $(0 < \alpha < 1)$. For $\beta \leq \alpha^{-1}$, let

$$R_\beta(\omega) = \{t : \limsup_{h \downarrow 0} h^{-\beta}(\tau(t) - \tau(t-h)) = \infty\} .$$

Then $\dim R_\beta \leq \alpha\beta$ a.s.

Proof. Let

$$C_n = \{[k2^{-n}, (k+2)2^{-n}] , k = 0,\dots,2^{n+1}\}$$

$$S = \{[s,t]: \tau(t) - \tau(s) > (t-s)^\beta\}$$

Now $P(\tau(1) > x) \leq cx^{-\alpha}$, and therefore

$$(3.1) \qquad P([s,t] \in S) = P(\tau(1) > (t-s)^{\beta-1/\alpha}) \qquad \text{(by scaling)}$$

$$\leq c(t-s)^{1-\alpha\beta} .$$

If $t \in R_\beta(\omega) \cap [0, 1]$, there exist $u_n \uparrow t$, and $k_n \uparrow \infty$, such that $\tau(t) - \tau(u_n) \geq k_n(t-u_n)^\beta$. We may take $k_n \geq 4^\beta$ for all n . Let m_n be such that $2^{-m_n-1} \leq t - u_n < 2^{-m_n}$: then there is an interval $[r_n, s_n]$ in C_{m_n} such that $[u_n, t] \subseteq [r_n, s_n]$. It follows that

$$\begin{aligned} \tau(s_n) - \tau(r_n) &\geq k_n(t-u_n)^\beta \\ &\geq k_n 4^{-\beta}(s_n-r_n)^\beta \\ &\geq (s_n-r_n)^\beta . \end{aligned}$$

Therefore each point in $R_\beta \cap [0, 1]$ is covered infinitely often by intervals in $\cup_m C_m \cap S$. Let N_m be the number of intervals in $C_m \cap S$, and $\gamma > \beta\alpha$: then by (3.1),

$$E \sum_m N_m 2^{-m\gamma} \le c \sum_m 2^{m+1} (2^{-m+1})^{1-\alpha\beta} 2^{-m\gamma}$$

$$= c\, 2^{2-\alpha\beta} \sum_m 2^{-m(\gamma-\alpha\beta)} < \infty .$$

Then $\dim R_\beta \cap [0, 1] \le \gamma$ and the result is now immediate. □

The above result is essentially due to Orey and Taylor [16]. Indeed it follows from the above and (6.3) of [16] that $\dim R_\beta = \alpha\beta$ a.s.

<u>Proposition 3.2</u>. $\dim \Lambda_p \le 1/4p$ <u>for all</u> $p \ge 1/2$.

<u>Proof</u>. Let $\hat{B}_t^{(r)} = B_r - B_{r-t}$ for $0 \le t \le r$, and $r \ge 0$, let $\hat{M}_t^{(r)} = \sup_{s \le t} \hat{B}_s^{(r)}$, and $S_r = \{t \le r : \hat{B}_t^{(r)} > \hat{B}_s^{(r)}$ for all $s < t\}$.

Then, if $q < p$,

$$\Lambda_p \subseteq \bigcup_{r \in \mathbb{Q}_+} \{t : r-t \in S_r , \text{ and } \liminf_{h \downarrow 0} h^{-q}(\hat{M}_{r-t}^{(r)} - \hat{M}_{r-t-h}^{(r)}) = 0\} .$$

It is therefore sufficient to show that, if $S = \{t \ge 0 : B_t > B_s$ for all $s \le t\}$, and $M_t = \sup_{s \le t} B_s$, and

$$A = \{t \in S: \liminf_{h \downarrow 0} h^{-q}(M_t - M_{t-h}) = 0\} ,$$

then $\dim A \le 1/4q$. The image of A under M is the set

$$M(A) = \{y \ge 0 : \limsup_{x \downarrow 0} x^{-1/q}(\tau(y) - \tau(y-x)) = \infty , \ y \text{ is a continuity point of } \tau\} ,$$

where $\tau_x = \inf\{t \ge 0 : M_t = x\}$. Now τ is a stable subordinator of index $\frac{1}{2}$, and so, by Lemma 3.1, $\dim M(A) \leq \frac{1}{2q}$ a.s. As $\tau(M_t) = t$ for all $t \in S$, we have $A = \tau(M(A))$. Hawkes and Pruitt, [8] show that, if Y is a stable subordinator of index $\beta < 1$, then, for all Borel sets B simultaneously, $\dim\{Y_t, t \in B\} = \beta \dim B$. Applying this theorem to $M(A)$ and τ, we have $\dim A = \frac{1}{2} \dim M(A) \leq \frac{1}{4q}$. Hence $\dim \Lambda_p \leq \frac{1}{4q}$, and letting $q \uparrow p$ we deduce the result. □

We now wish to obtain a lower bound on $\dim \Lambda_p(B)$. Let X be a Bes(3) process, $L_x = \sup\{t \geq 0 : X_t = x\}$, and

$$\Gamma_p = \{x : \text{there exists } t_n \downarrow 0 \text{ such that } X_{L_x+t_n} < x + t_n^p \text{ for all } n\}.$$

We begin by obtaining a lower bound on $\dim \Gamma_p$. The idea of the proof is to fix a set A in $[0, 1]$ of small dimension, and to attempt to apply the condensation argument of [16] on the set A. If $A \cap \Gamma_p \neq \emptyset$ for suitable sets A, then $\dim A + \dim \Gamma_p > 1$.

Let $0 < \alpha < 1$, $\delta_n = 2^{-n}$, and let A have the following properties:

(3.2) (a) There exists a sequence of finite sets A_n such that $A = c\ell(\bigcup_{n=1}^{\infty} A_n)$

(b) $|x-y| \geq \delta_n$ for all $x, y \in A_n$, $x \neq y$.

(c) For all $n \geq 1$, $a \in A_n$ and $\varepsilon > 0$, there exists $M = M(n,a,\varepsilon)$ such that $\#(A_m \cap (a, a+\varepsilon)) \geq \varepsilon\, \delta_m^{-\alpha}$ for all $m \geq M$.

(d) $A_n \subset (0,1)$ for all n.

For $n \geq 1$ let

$$B_n = \{x \in A_n : \text{there exists } 0 < t < L_{x+\delta_n} - L_x < (\log \tfrac{1}{\delta_n})^{-1} \text{ with } X_{L_x+t} - x < t^p\}.$$