Observations about two biquadratics, of which the sum is able to be resolved into two other biquadratics*

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1. Since it has been demonstrated for neither the sum nor the difference of two biquadratics to be able to be a square, much less will it be possible to be a biquadratic. As well, no less should trust be denied for the sum of three such biquadratics to ever be able to be a biquadratic, even if a demonstration of this has not been discovered. On the other hand, whether it is possible to find four biquadratics of which the sum is a biquadratic we may rightly be uncertain about, with no such biquadratics having yet been exhibited by anyone.

2. Although it would be possible to have demonstrated for no three biquadratics to be given of which the sum is also a biquadratic, it is however by no means permitted to extend this to differences, and therefore neither is it possible to be asserted for such an equation $A^4 + B^4 - C^4 = D^4$ to be impossible; in fact, I have observed for this equation to be able indeed to be resolved in infinitely many ways. Neither however do I intend to assert for

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this to have hitherto been exhibited by no one and now be vacant entirely; wholely, this memorial unravels this Analysis in general. I hope moreover for each of the methods which I have used not to be favored with attention altogether undeservedly. Moreover, it is clear for this to equally be turned to the question about two biquadratics, of which either their sums or their differences are mutually equal; for if it were $A^4 + B^4 = C^4 + D^4$, certainly it will then be $A^4 - D^4 = C^4 - B^4$, from which our Problem is proposed.

Problem

To find two biquadratics A^4 and B^4 of which it is possible to resolve the sum into two other biquadratics, so that such an equality may be obtained $A^4 + B^4 = C^4 + D^4$.

Solution

3. Since then it ought to be $A^4 - B^4 = C^4 - D^4$, we put

$$A = p + q;\ D = p - q;\ C = r + s \text{ and } B = r - s,$$

such that it follows that this equation is constructed

$$pq(pp + qq) = rs(rr + ss),$$

which would evidently be satisfied if it were to be taken $r = p$ and $s = q$, from which however I would clearly gain nothing, since from this the obvious case of $C = A$ and $B = D$ is born. At the same time though, this case may prevail to lead to other solutions.

4. We now put:

$$p = ax;\ q = by;\ r = kx \text{ and } s = y,$$

so that this equation which is to be resolved is obtained

$$ab(aaxx + bbyy) = k(kkxx + yy),$$

from which at once we deduce $\frac{yy}{xx} = \frac{k^3-a^3\cdot b}{db^3-k}$, and consequently this fraction of squares ought to be resolved. Here, the case presents itself immediately to the eyes, by use of which this will come, namely it is to be assumed $k = ab$, for then it will be

$$\frac{yy}{xx} = \frac{a^3b(bb-1)}{ab(bb-1)} = aa,$$

from which it will become $y = a, x = 1$ and then $p = a, q = ab, r = ab, s = a$, whose values follow easily from this case.

5. Then in the case of the following, we set $k = ab(1+z)$ and our equation is transformed into this form

$$\frac{yy}{xx} = \frac{a^3b((bb-1)+3bbz+3bbz^2+bbz^3)}{ab(bb-1-z)} = aa\frac{(bb-1+3bbz+3bbz^2+bbz^3)}{bb-1-z},$$

and from this equation we elicit

$$\frac{y}{x} = \frac{(a\surd(bb-1)^2+(3bb-1)(bb-1)z+3bb(bb-2)z^2+bb(bb-4)z^3bbz^4)}{bb-1-z}.$$

From this therefore, we bring the formula to a square:

$$(bb-1)^2+(bb-1)(3bb-1)z+3bb(bb-2)zz+bb(bb-4)z^3-bbz^4;$$

we set the root of it equal to

$$bb-1+fz+gzz,$$

and we assume the letters f and g such that the first three terms are destroyed. Since the square of this form will be:

$$(bb-1)^2+2(bb-1)fz+2(bb-1)gzz+2fgz^3+ggz^4+ffzz,$$

indeed the first of the terms will be destroyed spontaneously; so that likewise the second will turn out in the same way, it should be taken

$$f = \frac{3bb-1}{2},$$

and then for the third we will have

$$3bb(bb-2) = 2(bb-1)g + \frac{9b^4-6bb+1}{4},$$

from which it is gathered

$$g = \frac{3b^4 - 18bb - 1}{3(bb-1)};$$

with theses values having been defined, the equation which is to be resolved will be

$$(gg + bb)z = bb(bb - 4) - 2fg,$$

from which we gather

$$z = \frac{bb(bb-4) - 2fg}{bb + gg}.$$

6. So far, the letter b is relinquished to our discretion; therefore, it will be welcome for it to have been assumed, and hence simultaneously we will have determined the quantity z, and we will at once have

$$x = bb - 1 - z \text{ and } y = a(bb - 1 + fz + gzz),$$

and then in turn

$$\begin{aligned} p &= a(bb - 1 - z) \quad & r &= ab(1+z)(bb - 1 - z) \\ q &= ab(bb - 1 + fz + gzz) \quad & s &= a(bb - 1 + fz + gzz); \end{aligned}$$

since this formulas are all divisible by a, it will be permitted to remove this by division, so that it will be

$$\begin{aligned} p &= bb - 1 - z \quad & r &= b(1+z)(bb - 1 - z) \\ q &= b(bb - 1 + fz + gzz) \quad & s &= bb - 1 + fz + gzz, \end{aligned}$$

where it is to be noted that if the numbers x and y have a common factor, before it is to be removed by the earlier division, the letters p, q, r, s may be defined from it. For this work it will therefore be worthwhile to unfold some special solutions. However, it is indeed at once apparent for it not to be able to be taken $b = 1$, because it would be $g = \infty$, and indeed no less may it be put $b = 0$, because then it will be $q = 0$; from this, we obtain two cases, the first namely $b = 2$, then indeed $b = 3$.

1st special solution

7. Were it $b = 2$, the above values are assembled so that it follows:

$$f = \frac{11}{2};\ g = -\frac{25}{24};\ z = \frac{6600}{2929},$$

next, because the letter a clearly does not enter into the calculation, in its place is written unity; then indeed it will be

$$x = 3 - \frac{6600}{2929} = \frac{2187}{2929};\ y = 3 + \frac{11}{2} \cdot \frac{6600}{2929} - \frac{25}{24} \cdot \frac{6600^2}{2929^2} = \\ 3 + \frac{55407 \cdot 1100}{2929^2} = \frac{3 \cdot 28894941}{2929^2},$$

and the entire business is turned to the ratio between x and y, which because it is

$$\frac{y}{x} = \frac{3 \cdot 28894941}{2187 \cdot 2929} = \frac{28894941}{2929 \cdot 729} = \frac{3210549}{2929 \cdot 81} = \frac{1070183}{27 \cdot 2929},$$

we will have

$$x = 79083 \text{ and } y = 1070183,$$

then therefore, because

$$k = 2(1+z) = \frac{2 \cdot 9529}{2929} = \frac{19058}{2929}, \text{ we may conclude it to be}$$

$$p = 79083;\ r = 27 \cdot 19058 = 514566$$

$$q = 2 \cdot 1070183 = 2140366;\ s = 1070 \cdot 183.$$

Consequently, we obtain for the roots of these biquadratics

$$A = p + q = 2219449;\ C = 1584749$$

$$B = r - s = -555617;\ D = 2061283$$

and therefore it will be $A^4 + B^4 = C^4 + D^4$.

2nd special solution

8. Were it $b = 3$, it will be $f = 13$ and $g = \frac{5}{4}$, then $z = \frac{200}{169}$, and thus

$$k = \frac{3 \cdot 369}{169} = \frac{1107}{169} = \frac{9 \cdot 123}{169} = \frac{27 \cdot 41}{169}, \text{ in turn } x = \frac{8 \cdot 144}{169} = \frac{128 \cdot 9}{169} \text{ and}$$

$$y = 8 + \frac{200}{169}\left(13 + \frac{5}{4} \cdot \frac{200}{169}\right) = 8 + \frac{200}{169} \cdot \frac{2447}{169} = \frac{8 \cdot 150911}{169^2}$$