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THE

THEORY OF DETERMINANTS

IN THE

HISTORICAL ORDER OF ITS DEVELOPMENT

PART I.

DETERMINANTS IN GENERAL

LEIBNITZ (1693) TO CAYLEY (1841)

BY

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PREFACE.

DURING the writing of my *Treatise on the Theory of Determinants* (Macmillan & Co., 1882), it was repeatedly forced on my attention that the history of the subject had been very imperfectly looked into. Not only, as it appeared, had injustice been done by the attribution of isolated theorems and demonstrations to authors other than the first discoverers, but the labours of the great founders of the theory had been disproportionately represented, and a considerable amount of valuable work had actually been lost sight of altogether. This state of matters, it was clear, had its explanation in the fact that two of the foremost nationalities of Europe had not contributed their proper shares to the work of historical research. France, for some reason or other, had taken comparatively slight interest in any part of the subject; and England, though interested in the *theory* and contributing largely to its development, had been content as regards the *history* simply to accept and promulgate the results of German assiduity.* Not unnaturally, therefore, mathematicians in general had come to look at the history almost as it were through German spectacles. It could not be said, moreover, that this optical help was detri-

* The apathy of France seems the more blameworthy, because, as will appear, the theory in its early stages was essentially a creation of French genius. As for England, signs have not been awanting during recent years that she means for the future to take her part in the investigation and exposition of mathematical history.

mental to authors outside Germany and to them alone, for several German mathematicians of note had not had justice done them, their writings * in one or two instances being even unknown in the land that produced them.

With these facts before me, I resolved to set about collecting the whole literature of the subject, in order that, as a first step on the way to a history, a *bibliography* might be compiled. To those who know how lamentably ill-provided mathematicians are with guides to their literature, it will readily occur that this initiatory step entailed a vast amount of labour. When so far completed, the "List of Writings," as it was called, was published in the *Quarterly Journal of Mathematics*, of which it occupies 41 pages.† Its reception was all that could be desired. Reward for the trouble it had cost soon came to hand in the form of addenda from many widely separated correspondents, two of whom, I cannot but recall, examined the scientific serials of their own countries in order to check and supplement the list. An additional list, extending to 22 pages,‡ thus came to be published, and with its publication the preparatory stage was deemed to be over.

The method followed in using the material of the two lists to produce a work, intended to elucidate Determinants by showing the theory in the actual process of growth, is explained in the Introduction (pp. 2–5).

My object, it may be added, has been twofold. First, to provide a work of reference which should contain all that had been written on the subject, and which should be so indexed that any one engaged in research might easily ascertain exactly what had

* Notably that of Schweins. See an article in the *Philosophical Magazine*, vol. xviii. pp. 416–427 (1884), entitled "An Overlooked Discoverer in the Theory of Determinants."

† Vol. xviii. pp. 110–149.

‡ Vol. xxi. pp. 299–320.

been done on any particular topic, how it had been done, and what possible developments it foreshadowed. Secondly, to show clearly to whom every step in advance had been due, doing this in such a way, also, that the reader might see the actual data on which any conclusion was based. Knowing the value of the historical method as a means of *teaching* any branch of science, I cannot but hope that a third result may follow in at least a certain small measure —viz., that some who have never studied Determinants at all may thus readily and in an interesting manner acquire a knowledge of what on all hands is conceded to be a singularly beautiful department of analysis.

My warmest acknowledgments are due to the Members of Council of the Royal Society of Edinburgh, without whose encouragement the work would probably not have been undertaken, and without whose aid it would certainly not have appeared in its present connected form.

T. M.

BEECHCROFT, BOTHWELL,
GLASGOW, *Feb. 15, 1890.*

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b

HISTORY

OF THE

THEORY OF DETERMINANTS.

Part I.—*Determinants in General.*

In October 1881 I published in the *Quarterly Journal of Mathematics* (xviii. pp. 110–149) a "List of Writings on Determinants," which contained the titles of all the books, pamphlets, memoirs, magazine articles, &c., which were then known to me to exist on the subject of the Theory of Determinants. The list consisted of 489 entries arranged in chronological order, the first date being 1693, and the last 1880. During the three years which have elapsed since it was published, I have been steadily making manuscript additions to it, not merely in the way of continuation for the purpose of keeping it up to date, but also by the intercalation of omitted titles unearthed in the course of my own researches, or brought to my notice by obliging correspondents.

The continuation of the list from 1880 forwards is comparatively an easy matter: it is not by any means easy to render equally complete that portion of the list which pertains to the eighteenth century. In the early history of a scientific subject, before the nomenclature has become fixed, the mere *titles* of writings are insufficient guides: the searcher's work is, consequently, minute and laborious, and he never can be quite sure that his labours are at an end. As far, however, as Determinants are concerned, I am inclined now to think that the writings which are unknown cannot be of much importance, and that the time has come for using the collected material in the production of a detailed history of the subject.

The plan proposed to be followed is not to give one connected history of determinants as a whole, but to give separately the history of each of the sections into which the subject has been divided, viz., to deal with determinants in general, and thereafter in order with the various special forms. This will not only tend to smoothness in the narrative by doing away with the necessity of frequent harkings back, but it will also be of material importance to investigators who may wish to find out what has already been done in advancing any particular department of the subject. To this end, also, each new result as it appears will be numbered in Roman figures; and if the same result be obtained in a different way, or be generalised, by a subsequent worker, it will be marked among the contributions of the latter with the same Roman figures, followed by an Arabic numeral. Thus the theorem regarding the effect of the transposition of two rows of a determinant will be found under Vandermonde, marked with the number xi., and the information intended thus to be conveyed is that in the order of discovery the said theorem was the *eleventh* noteworthy result obtained: while the mark XI. 2, which occurs under Laplace, is meant to show that the theorem was not then heard of for the first time, but that Laplace contributed something additional to our knowledge of it. In this way any reader who will take the trouble to look up the sequence XI., XI. 2, XI. 3, &c., may be certain, it is hoped, of obtaining the full history of the theorem in question.

The early foreshadowings of a new domain of science, and tentative gropings at a theory of it, are so difficult for the historian to represent without either conveying too much or too little, that the only satisfactory way of dealing with a subject in its earliest stages seems to be to reproduce the exact words of the authors where essential parts of the theory are concerned. This I have resolved to do, although to some it may have the effect of rendering the account at the commencement somewhat dry and forbidding.

No author, so far as I am aware, has preceded me in the task I have chosen. Sketches of the history have appeared in a number of text-books of the subject, notably in Günther's *Lehrbuch der Determinanten-Theorie für Studirende* (2^te Aufl. xii. 209 pp., Erlangen, 1877), which contains a considerable quantity of detail. The *early* history has been very carefully dealt with by F. J. Studnička, in a memoir published in the *Abhandlungen der königl. böhm. Gesellschaft der Wissenschaften*, 6 Folge, viii. 40 pp. (24th

March 1876), and entitled "A. L. Cauchy als formaler Begründer der Determinanten-Theorie. Eine literarisch-historische Studie." There is also an academic thesis (*Teorin för Determinant-Kalkylen*, 121 pp., Helsingfors, 1st March 1876), by E. J. Mellberg, which treats somewhat at length of the early authorities. The existence of these two latter writings has not, however, induced me to curtail to any extent the corresponding part of my work.

LEIBNITZ (1693).

[Leibnizen's mathematische Schriften, herausg. v. C. I. Gerhardt 1 Abth. ii. pp. 229, 238–240, 245, Berlin, 1850.]

In the fourth letter of the published correspondence between Leibnitz and De L'Hospital, the former incidentally mentions that in his algebraical investigations he occasionally uses numbers instead of letters, treating the numbers however as if they were letters. De L'Hospital, in his reply, refers to this, stating that he has some difficulty in believing that numbers can be as convenient or give as general results as letters. Thereupon Leibnitz, in his next letter (28th April 1693), proceeds with an explanation:—

> "Puisque vous dites que vous avés de la peine à croire qu'il soit aussi general et aussi commode de se servir des nombres que des lettres, il faut que je ne me sois pas bien expliqué. On ne sçauroit douter de la generalité en considerant qu'il est permis de se servir de 2, 3, etc., comme d' *a* ou de *b*, pour veu qu'on considere que ce ne sont pas de nombres veritables. Ainsi 2.3 ne signifie point 6 mais autant qu' *ab*. Pour ce qui est de la commodité, il y en a des très grandes, ce qui fait que je m'en sers souvent, sur tout dans les calculs longs et difficiles ou il est aisé de se tromper. Car outre la commodité de l'épreuve par des nombres, et même par l'abjection du novenaire, j' y trouve un tres grand avantage même pour l'avancement de l'Analyse. Comme c'est une ouverture assez extraordinaire, je n'en ay pas encor parlé à d'autres, mais voicy ce que c'est. Lorsqu'on a besoin de beaucoup de lettres, n'est il pas vray que ces lettres n'expriment point les rapports qu'il y a entre les grandeurs qu'elles signifient, au lieu qu'en me servant des nombres je puis exprimer ce rapport. Par exemple soyent

proposées trois equations simples pour deux inconnues à dessein d'oster ces deux inconnues, et cela par un canon general. Je suppose

$$10 + 11x + 12y = 0 \qquad (1)$$
$$\text{et} \qquad 20 + 21x + 22y = 0 \qquad (2)$$
$$\text{et} \qquad 30 + 31x + 32y = 0 \qquad (3)$$

ou le nombre feint estant de deux characteres, le premier me marque de quelle equation il est, le second me marque à quelle lettre il appartient. Ainsi en calculant on trouve par tout des harmonies qui non seulement nous servent de garans, mais encor nous font entrevoir d'abord des regles ou theoremes. Par exemple ostant premierement y par la premiere et la seconde equation, nous aurons:

$$\begin{matrix} +10 \,.\, 22 + 11 \,.\, 22x \\ -12 \,.\, 20 - 12 \,.\, 21 \,.\,. \end{matrix} = 0 \qquad (4)^*$$

et par la premiere et troisieme nous aurons:

$$\begin{matrix} +10 \,.\, 32 + 11 \,.\, 32x \\ -12 \,.\, 30 - 12 \,.\, 31 \,.\,. \end{matrix} = 0 \qquad (5)$$

ou il est aise de connoistre que ces deux equations ne different qu'en ce que le charactere antecedent 2 est changé au charactere antecedent 3. Du reste, dans un même terme d'une même equation les characteres antecedens sont les mêmes, et les characteres posterieurs font une même somme. Il reste maintenant d'oster la lettre x par la quatrieme et cinquieme equation, et pour cet effect nous aurons †

$$\begin{matrix} 1_0 \,.\, 2_1 \,.\, 3_2 \\ 1_1 \,.\, 2_2 \,.\, 3_0 \\ 1_2 \,.\, 2_0 \,.\, 3_1 \end{matrix} \quad = \quad \begin{matrix} 1_0 \,.\, 2_2 \,.\, 3_1 \\ 1_1 \,.\, 2_0 \,.\, 3_2 \\ 1_2 \,.\, 2_1 \,.\, 3_0 \end{matrix}$$

qui est la derniere equation delivrée des deux inconnues qu'on vouloit oster, et qui porte sa preuve avec soy par les harmonies qui se remarquent par tout, et qu'on auroit bien de la peine à decouvrir en employant des lettres a, b, c, sur tout

* This is written shortly for $\left.\begin{matrix} +10 \,.\, 22 + 11 \,.\, 22x = 0 \\ -12 \,.\, 20 - 12 \,.\, 21x = 0 \end{matrix}\right\}$

† The author here slightly changes his notation. What is meant to be is indicated

$$10 \,.\, 21 \,.\, 32 + 11 \,.\, 22 \,.\, 30 + 12 \,.\, 20 \,.\, 31 = 10 \,.\, 22 \,.\, 31 + 11 \,.\, 20 \,.\, 32 + 12 \,.\, 21 \,.\, 30.$$

lors que le nombre des lettres et des equations est grand. Une partie du secret de l'analyse consiste dans la caracteristique, c'est à dire dans l'art de bien employer les notes dont on se sert, et vous voyés, Monsieur, par ce petit echantillon, que Viete et des Cartes n'en ont pas encor connu tous les mysteres. En poursuivant tant soit peu ce calcul on viendra à un *theoreme general* pour quelque nombre de lettres et d'equations simples qu'on puisse prendre. Le voicy comme je l'ay trouvé autres fois :

"*Datis aequationibus quotcunque sufficientibus ad tollendas quantitates, quae simplicem gradum non egrediuntur, pro aequatione prodeunte, primo sumendae sunt omnes combinationes possibiles, quas ingreditur una tantum coefficiens uniuscujusque aequationis : secundo, eae combinationes opposita habent signa, si in eodem aequationis prodeuntis latere ponantur, quae habent tot coefficientes communes, quot sunt unitates in numero quantitatum tollendarum unitate minuto : caeterae habent eadem signa.*

"J'avoue que dans ce cas des degrés simples on auroit peut estre decouvert le même theoreme en ne se servant que de lettres à l'ordinaire, mais non pas si aisement, et ces adresses sont encor bien plus necessaires pour decouvrir des theoremes qui servent à oster les inconnues montées à des degrés plus hauts. Par exemple, ... "

It will be seen that what this amounts to is *the formation of a rule for writing out the resultant of a set of linear equations.* When the problem is presented of eliminating x and y from the equations

$$a+bx+cy=0\,, \qquad d+ex+fy=0\,, \qquad g+hx+ky=0,$$

Leibnitz in effect says that first of all he prefers to write 10 for a, 11 for b, and so on; that, having done this, he can all the more readily take the next step, viz., forming every possible product whose factors are one coefficient from each equation,* the result being

$$\begin{array}{ccc} 10.21.32, & 10.22.31, & 11.20.32, \\ 11.22.30, & 12.20.31, & 12.21.30; \end{array}$$

and that, then, *one* being the number which is less by one than the

* Of course, this is not exactly what Leibnitz meant to say.

number of unknowns, he makes those terms different in sign which have only *one* factor in common.

The contributions, therefore, which Leibnitz here makes to algebra may be looked upon as three in number :—

(1) A *new notation*, numerical in character and appearance, for individual members of an arranged group of magnitudes; the two numbers which constitute the notation being like the Cartesian co-ordinates of a point in that they denote any one of the said magnitudes by indicating its position in the group, . . . (I.)

(2) A rule for *forming the terms* of the expression which equated to zero is the result of eliminating the unknowns from a set of simple equations, ... (II.)

(3) A rule for *determining the signs* of the terms in the said result. (III.)

The last of these is manifestly the least satisfactory. In the first place, part of it is awkwardly stated. Making those terms different in sign *which have only as many factors alike as is indicated by the number which is less by one than the number of unknown quantities* is exactly the same as making those terms different in sign *which have only two factors different.* Secondly, in form it is very unpractical. The only methodical way of putting it in use is to select a term and make it positive; then seek out a second term, having all its factors except two the same as those of the first term, and make this second term negative; then seek out a third term, having all its factors except two the same as those of the second term, and make this third term positive; and so on.

Although there is evidence that Leibnitz continued, in his analytical work, to use his new notation for the coefficients of an equation (see Letters xi., xii., xiii. of the said correspondence), and that he thought highly of it (see Letter viii. "chez moi c'est une des meilleures ouvertures en Analyse"), it does not appear that by using it in connection with sets of linear equations, or by any other means, he went further on the way towards the subject with which we are concerned. Moreover, it must be remembered that the little he did effect had no influence on succeeding workers. So far as is known, the passage above quoted from his correspondence with De L'Hospital was not published until 1850. Even for some little time after the date of Gerhardt's publication it escaped observation, Lejeune Dirichlet being the first to note its historical importance. It is true that during his own lifetime, Leibnitz's *use of numbers in place of letters* was made known to the world in the *Acta Eruditorum* of Leipzig for the year 1700 (*Responsio ad Dn. Nic. Fatii Duillerii imputationes*, pp. 189–208); but the particular application of the new symbols which brings them into connection with determinants was not there given.

CRAMER (1750).

[Introduction a l'Analyse des Lignes Courbes algébriques, par Gabriel Cramer, pp. 59, 60, 656–659. Génève, 1750.]

The third chapter of Cramer's famous treatise deals with the different *orders* (degrees) of curves, and one of the earliest theorems of the chapter is the well-known one that the equation of a curve of the nth degree is determinable when $\frac{1}{2}n(n+3)$ points of the curve are known. In illustration of this theorem he deals (p. 59) with the case of finding the equation of the curve of the *second* degree which passes through *five* given points. The equation is taken in the form

$$A + By + Cx + Dyy + Exy + xx = 0\ ;$$

the five equations for the determination of A, B, C, D, E are written down; and it is pointed out that all that is necessary is the solution of the set of five equations, and the substitution of the values of A, B, C, D, E thus found. "Le calcul véritablement en seroit assez long," he says; but in a footnote there is the remark that it is to algebra we must look for the means of shortening the process, and we are directed to the appendix for a convenient general rule which he had discovered for obtaining the solution of a set of equations of this kind. The following is the essential part of the passage in which the rule occurs :—

> "Soient plusieurs inconnues z, y, x, v, &c., et autant d'équations
>
> $$A^1 = Z^1z + Y^1y + X^1x + V^1v + \&c.$$
> $$A^2 = Z^2z + Y^2y + X^2x + V^2v + \&c.$$
> $$A^3 = Z^3z + Y^3y + X^3x + V^3v + \&c.$$
> $$A^4 = Z^4z + Y^4y + X^4x + V^4v + \&c.$$
> &c.

où les lettres A^1, A^2, A^3, A^4, &c., ne marquent, pas comme à l'ordinaire, les puissances d' A, mais le prémier membre, supposé connu, de la prémiére, seconde, troisiéme, quatriéme, &c. équation."

[Here the solutions of the cases of 1, 2, and 3 unknowns are given, and he then proceeds.]

" L'examen de ces Formules fournit cette Règle générale. Le nombre des équations et des inconnues étant *n*, on trouvera la valeur de chaque inconnue en formant *n* fractions dont le dénominateur commun à autant de termes qu'il y a de divers arrangements de *n* choses différentes. Chaque terme est composé des lettres ZYXV, &c., toujours écrites dans le même ordre, mais auxquelles on distribue, comme exposants, les *n* prémiers chiffres rangés en toutes les maniéres possibles. Ainsi, lorsqu'on a trois inconnues, le dénominateur a [$1 \times 2 \times 3 =$] 6 termes, composés des trois lettres ZYX, qui recoivent successivement les exposants 123, 132, 213, 231, 312, 321. On donne à ces termes les signes + ou −, selon la Règle suivante. Quand un exposant est suivi dans le même terme, médiatement ou immédiatement, d'un exposant plus petit que lui, j'appellerai cela un *dérangement.* Qu'on compte, pour chaque terme, le nombre des dérangements : s'il est pair ou nul, le terme aura le signe + ; s'il est impair, le terme aura le signe −. Par ex. dans le terme $Z^1Y^2V^3$ il n'y a aucun dérangement ; ce terme aura donc le signe +. Le terme $Z^3Y^1X^2$ a aussi le signe +, parce qu'il a deux dérangements, 3 avant 1 et 3 avant 2. Mais le terme $Z^3Y^2X^1$, qui a trois dérangements, 3 avant 2, 3 avant 1, et 2 avant 1, aura le signe −.

"Le dénominateur commun étant ainsi formé, on aura la valeur de *z* en donnant à ce dénominateur le numérateur qui se forme en changeant, dans tous ces termes, Z en A. Et la valeur d' *y* est la fraction qui a le même dénominateur et pour numérateur la quantité qui résulte quand on change Y en A, dans tous les termes du dénominateur. Et on trouve d'une maniére semblable la valeur des autres inconnues."

It is evident at once that the new results here given are—

(1) A rule for *forming the terms* of the common denominator of

the fractions which express the values of the unknowns in a set of linear equations, ... (IV.)

(2) A rule for *determining the sign* of any individual term in the said common denominator (and, included in the rule, the notion of a "dérangement"), ... (III. 2)

(3) A rule for *obtaining the numerators* from the expression for common denominator, ... (V.)

The problem which Cramer set himself at this point in his book was exactly that which Leibnitz had solved, viz., the elimination of n quantities from a set of $n+1$ linear equations. The solution which Cramer obtained, and which, be it remarked, was the solution best adapted for his purpose, was quite distinct in character from that of Leibnitz. Leibnitz gave a rule for writing out the final result of the elimination; what Cramer gives is a rule for writing out the values of the n unknowns as determined from n of the $n+1$ equations, after which we have got to substitute these values in the remaining $(n+1)$th equation. The notable point in regard to the two solutions is, that Cramer's rule for writing the *common denominator* of the values of the n unknowns (an expression of the nth degree in the coefficients) is exactly Leibnitz's rule for writing the *final result*, which is an expression of the $(n+1)$th degree. Had either discoverer been aware that the same rule sufficed for obtaining both of these expressions, he could not have failed, one would think, to note the *recurrent* law of formation of them. The result of eliminating w, x, y, z from the equations,

$$a_rw + b_rx + c_ry + d_rz = e_r \quad (r = 1, 2, 3, 4, 5)$$

is, according to Leibnitz, if we embody his rule in a later symbolism,

$$|\, a_1b_2c_3d_4e_5 \,| = 0\,;$$

whereas, according to Cramer, it is—

$$a_1\frac{|e_2b_3c_4d_5|}{|a_2b_3c_4d_5|} + b_1\frac{|a_2e_3c_4d_5|}{|a_2b_3c_4d_5|} + c_1\frac{|a_2b_3e_4d_5|}{|a_2b_3c_4d_5|} + d_1\frac{|a_2b_3c_4e_5|}{|a_2b_3c_4d_5|} = e_1,$$

and from the collocation of these the one natural step is to the identity

$$-|a_1b_2c_3d_4e_5| = a_1|e_2b_3c_4d_5| + b_1|a_2e_3c_4d_5| + \ldots\ldots - e_1|a_2b_3c_4d_5|\,.$$

The fate of Cramer's rule was very different from that of Leibnitz.