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CLONES IN UNIVERSAL ALGEBRA

ÁGNES SZENDREI
József Attila University
Szeged

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FOREWORD

The investigation of clones originates partly in logic, namely in the study of composition of truth functions, and partly in universal algebra, from the observation that most properties of algebras depend on their term operations rather than on the choice of their basic operations. During the last fifteen years or so the combination of these two aspects and the application of new algebraic methods produced a rapid development, and by now the theory of clones has become an integral part of universal algebra.

The aim of these lecture notes is to introduce the reader to some results showing how clones can contribute to the understanding of the structure of algebras, and not less importantly, to present several techniques in clone theory. To keep the length within reasonable limits I had to select a few topics. The choice is certainly rather subjective. I took this opportunity to reconsider a number of results I was interested in, and to point out some connections between them, a part of which may not have been known before.

The book is self-contained, the reader is assumed to be familiar only with the rudiments of universal algebra and lattice theory, and some basic facts in other fields of abstract algebra (e.g. groups, permutation groups, rings, modules). Four textbooks, three on universal algebra and one on clones, are listed on p. 159. Throughout, they are referred to by an abbreviation. To all other books and papers reference is made by the author's name and year of publication.

This volume is an extended version of my lectures delivered at the 23rd Session of the Séminaire de mathématiques supérieures on "Universal Algebra and Relations", held at the Université de Montréal in 1984. I wish to express my thanks to the organizers, Professors P. Berthiaume and I.G. Rosenberg, for the invitation. I am greatly indebted to P.P. Pálfy who read most of the manuscript and made a lot of valuable suggestions and corrections. I am grateful also for the helpful remarks by B. Csákány and L. Szabó.

CONTENTS

Chapter 1

ALGEBRAS AND CLONES

The notion of a clone, which will be fundamental throughout these notes, was introduced by P. Hall. In 2-valued as well as in multiple-valued logic a related concept, called closed class, or iterative class of truth functions was used already by E.L. Post [1921], [1941]. However, the importance of clones in universal algebra was not recognized until the early seventies.

By an operation we will always mean a finitary, nonnullary operation. (The exclusion of nullary operations does not cause any essential restriction in generality.) Let A be a set. For integers $n \geq 1$ and $1 \leq i \leq n$, the *i-th n-ary projection* on A is the operation defined by

$$e_{n,i}(a_1,\ldots,a_n) = a_i \quad \text{for all} \quad a_1,\ldots,a_n \in A.$$

If f is an n-ary and $g_1,\ldots,g_n$ are k-ary operations on A, then we define a k-ary operation $f(g_1,\ldots,g_n)$ on A, called the *superposition* of $f,g_1,\ldots,g_n$, as follows:

$$f(g_1,\ldots,g_n)(a_1,\ldots,a_k) = f(g_1(a_1,\ldots,a_k),\ldots,g_n(a_1,\ldots,a_k))$$

for all $a_1,\ldots,a_k \in A$.

DEFINITION. A set of operations on a fixed set A is said to be a *clone* on A iff it contains the projections and is closed under superposition.

Clearly, the set O_A of all operations on A, and the set J_A of all projections on A are clones. Furthermore, the intersection of arbitrary set of clones on A is again a clone. Thus it follows that the clones on A form a complete lattice Lat(A) with least element J_A and greatest element O_A. Furthermore, for arbitrary set F of operations on A there exists a least clone containing F. As usual, this clone will be called the *clone generated by* F, and will be denoted by [F]. Instead of [{f}] we write simply [f]. For a clone C and $n \geq 1$ we let $C^{(n)}$ denote the set of n-ary operations from C.

An n-ary operation f on A is said to *depend* on its i-th variable $(1 \leq i \leq n)$ iff there exist elements $a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_n,b,c \in A$ such that

$$f(a_1,\ldots,a_{i-1},b,a_{i+1},\ldots,a_n) \neq f(a_1,\ldots,a_{i-1},c,a_{i+1},\ldots,a_n).$$

Otherwise the i-th variable of f is called *fictitious*. In that case f can be regarded to arise from an (n-1)-ary operation by adding a new, fictitious variable in the i-th place. Using the projections it is easy to see that for every clone C and every operation $g \in C$, C contains all operations arising from g by identifying variables, or by permuting variables, or by adding fictitious variables.

Two clones are naturally attached to every algebra $\mathfrak{A} = (A;F)$: the clone $T(\mathfrak{A})$ of term operations (or term functions [BS], polynomials [G]) of $\mathfrak{A}$, which is the clone generated by F, and is called the *clone of* $\mathfrak{A}$; and the clone $P(\mathfrak{A})$ of polynomial operations (polynomials [BS], algebraic functions [G]) of $\mathfrak{A}$, which is the clone generated by F and all the unary constant operations on A.

EXAMPLES. 1. If $\underline{A} = (A;+,-,0)$ is an Abelian group, then

$$T(\underline{A}) = \{\sum_{i=1}^{n} c_i x_i : n \geq 1,\ c_1,\ldots,c_n \in \mathbf{Z}\},$$

$$\mathcal{P}(\underline{A}) = \{\sum_{i=1}^{n} c_i x_i + a: n \geq 1,\ c_1,\ldots,c_n \in \mathbf{Z},\ a \in A\}$$

($\mathbf{Z}$ denotes the set of integers).

2. For a ring R with 1 and a unitary R-module ${}_R\underline{A} = (A;+,-,\mathbf{0},R)$ we have

$$\mathcal{T}({}_R\underline{A}) = \{\sum_{i=1}^{n} r_i x_i: n \geq 1,\ r_1,\ldots,r_n \in R\},$$

$$\mathcal{P}({}_R\underline{A}) = \{\sum_{i=1}^{n} r_i x_i + a: n \geq 1,\ r_1,\ldots,r_n \in R,\ a \in A\}.$$

3. For a commutative ring R with 1, the clone $\mathcal{P}(R)$ consists of all polynomial functions of several variables (in the classical sense) on R.

4. Lagrange's interpolation theorem for functions of several variables implies that $\mathcal{P}(K) = \mathcal{O}_K$ for every finite field K.

5. Let $\mathcal{B}_2 = (\{0,1\};\wedge,\vee,r,0,1)$ be the 2-element Boolean algebra (r stands for complementation). It is known from elementary logic that $\mathcal{T}(\mathcal{B}_2) = \mathcal{O}_{\{0,1\}}$.

DEFINITION. Two algebras with a common universe are called *term equivalent* [*polynomially equivalent*] iff they have the same term [polynomial] operations. We say that two algebras $\mathfrak{A}_i = (A_i;F_i)$ $(i = 1,2)$ are *equivalent* iff $\mathfrak{A}_1$ is isomorphic to an algebra term equivalent to $\mathfrak{A}_2$.

It is easy to see that all three relations defined above are indeed equivalence relations. Equivalence of algebras is just the abstract version of term equivalence. Term equivalence is important because term equivalent algebras behave very similarly: they have the same subalgebras, endomorphisms, congruences, etc., moreover, they have finite bases for their identities simultaneously provided both are of finite type. In fact, most properties of an algebra depend on its clone rather than the choice of its basic operations. Several well-known instances of term equivalence are listed below to illustrate that term equivalent algebras are indeed "essentially the same".

EXAMPLES. 1. A $\mathbf{Z}$-module and its additive group;

2. a Boolean algebra and the corresponding Boolean ring with 1;

3. a zero ring and its additive group are term equivalent.

Recall that an operation f on A is *idempotent* iff it satisfies the identity $f(x,\dots,x) = x$. A clone on A is called idempotent iff it consists of idempotent operations, while an algebra $(A;F)$ is idempotent iff its basic operations (or equivalently, its term operations) are idempotent.

DEFINITION. Let $\mathfrak{A}_1 = (A;F_1)$ be an algebra. An algebra $\mathfrak{A}_2 = (A;F_2)$ is said to be a *reduct* [*polynomial reduct*] of $\mathfrak{A}_1$ iff $T(\mathfrak{A}_2) \subseteq T(\mathfrak{A}_1)$ [$P(\mathfrak{A}_2) \subseteq P(\mathfrak{A}_1)$, respectively]. The *full idempotent reduct* of $\mathfrak{A}_1$ is the algebra on A whose operations are the idempotent term operations of $\mathfrak{A}_1$.

EXAMPLE. The full idempotent reduct of a unitary R-module ${}_R\underline{A} = (A;+,-,0,R)$ is the algebra

$$(A;\{\sum_{i=1}^{n} r_i x_i \colon n \geq 1,\ r_1,\dots,r_n \in R,\ \sum_{i=1}^{n} r_i = 1\}),$$

which is term equivalent to the affine R-module

$$(A;x-y+z,\{rx + (1-r)y \colon r \in R\})$$

corresponding to ${}_R\underline{A}$ (cf. Proposition 2.6).

It is easy to see that a clone C on A is the clone of a unary algebra with universe A if and only if every operation in C depends on at most one of its variables. These clones will be termed *unary clones*. Occasionally it will be convenient to call J_A the *trivial clone* on A. Accordingly, by a *trivial algebra* we mean an algebra whose basic operations (or equivalently, term operations) are projections.

For an algebra $\mathfrak{A} = (A;F)$ a subset B of A is said to be a *subuniverse* of $\mathfrak{A}$ iff B is empty or is the universe of a subalgebra of $\mathfrak{A}$.

The fundamental notions of universal algebra and lattice theory, which are not defined here, can be found in any textbook on universal algebra, say [BS], [G], or [MMPT]. The 1-element algebras are considered simple. For a set A, the equality relation $\Delta_A = \{(a,a): a \in A\}$ and the total relation $\nabla_A = A^2$ are called *trivial equivalence relations* on A. The identity mapping $A \to A$ will be denoted by id_A. (The subscripts A can be omitted if there is no danger of confusion.) For an Abelian group $\underline{A} = (A;+,-,0)$ the *exponent* of $\underline{A}$ is the least positive integer n such that $\underline{A}$ satisfies the identity $nx = 0$, if such an n exists; otherwise the exponent of $\underline{A}$ is 0.

Term operations and subalgebras

The most fundamental observation in clone theory is that the clone $T(\mathfrak{A})$ of an algebra $\mathfrak{A} = (A;F)$ can also be described by "invariants" rather than its generating set F. Namely, these invariants are the subuniverses of powers of $\mathfrak{A}$. This connection between the term operations and the subuniverses of powers of $\mathfrak{A}$ is especially nice if $\mathfrak{A}$ is finite, for then the subuniverses of finite powers of $\mathfrak{A}$ already determine $T(\mathfrak{A})$. Although in a completely different terminology, these ideas go back to the investigations of A. V. Kuznetsov and his school, and M. Krasner in the forties and fifties. However, a systematic treatment was not available until the late sixties and early seventies.

The aim of this section is to record those facts from the general theory which will be needed later on. A more complete discussion can be found in the books [PK] and [MMPT; Chapter VII].

DEFINITION. Let A be a set and B a subset of a power A^I of A. An operation f on A is said to *preserve* B iff B is a subuniverse of the algebra $(A;f)^I$.

For example, if $B \subseteq A^2$ is an equivalence relation on A, then f preserves B means that B is a congruence of the algebra $(A;f)$, while if $B \subseteq A^2$ is a partial order on A, then f preserves B is equivalent to saying that f is monotone with respect to B.

The set of those operations on A which preserve a subset B of a power of A will be denoted by $\mathrm{Pol}_A\{B\}$ (as these operations are sometimes called polymorphisms). More generally, for arbitrary set S of subsets of powers of A we define the set $\mathrm{Pol}_A S$ of operations by

$$\mathrm{Pol}_A S = \bigcap_{B \in S} \mathrm{Pol}_A\{B\}.$$

(The subscript A may be omitted if it is understood from the context.)

To every function $f\colon A_1 \times \dots \times A_n \to A_o$ $(n \geq 1,\ A_o, A_1, \dots, A_n \subseteq A)$ we can assign in a natural way a subset of A^{n+1} as follows:

$$f^{\square} = \{(a_1, \dots, a_n, f(a_1, \dots, a_n)) : a_1 \in A_1, \dots, a_n \in A_n\}.$$

Sometimes it will be more convenient to write the values of f in the first component, that is, to associate with f the set

$$f_{\square} = \{(f(a_1, \dots, a_n), a_1, \dots, a_n) : a_1 \in A_1, \dots, a_n \in A_n\}$$

instead of $f^{\square}$. In most cases these notations will be used for operations.

DEFINITION. For two operations f, g on A we say that f *commutes* with g iff f preserves $g^{\square}$ (or equivalently, $g_{\square}$).

In particular, if g is unary, this means that g is an endomorphism of the algebra $(A;f)$. As we shall see below (Proposition 1.1(b)), the commutativity of two operations is a symmetric relation.

The following simple facts are immediate consequences of the definitions, therefore the proofs are left to the reader.

PROPOSITION 1.1. *Let* A *be a set.*

(a) $\mathrm{Pol}_A\, S$ *is a clone for arbitrary set* S *of subsets of powers of* A.

(b) *An operation* $f \in \mathcal{O}_A^{(k)}$ *commutes with an operation* $g \in \mathcal{O}_A^{(n)}$ *if and only if* f *and* g *satisfy the identity*

$$f(g(x_{11},\ldots,x_{1n}),\ldots,g(x_{k1},\ldots,x_{kn})) = g(f(x_{11},\ldots,x_{k1}),\ldots,f(x_{1n},\ldots,x_{kn})).$$

(c) *For a subset* B *of* A *and for arbitrary operations* $f \in \mathcal{O}_A$, $h \in \mathcal{O}_B$, *we have* $f \in \mathrm{Pol}_A\{h^{\square}\}$ *if and only if* B *is closed under* f *and the restriction* $f|_B$ *of* f *commutes with* h.

Let us consider a set T of n-ary operations on A. Each member of T is a mapping $A^n \to A$, and hence is an element of A^{A^n}. Thus T corresponds to a subset X_T of A^{A^n}. The proof of the following lemma is again straightforward.

LEMMA 1.2. *Let* T *be a set of* n-*ary operations on* A. *A* k-*ary operation* g *on* A *preserves* X_T *if and only if for all operations* $f_1,\ldots,f_k \in T$ *we have* $g(f_1,\ldots,f_k) \in T$.

Now we can state more precisely the connection between the term operations and the subuniverses of powers of an algebra.

PROPOSITION 1.3. *Let* $\mathfrak{A} = (A;F)$ *be an algebra. For every integer* $n \geq 1$, $X_{T^{(n)}(\mathfrak{A})}$ *is a subuniverse of* $\mathfrak{A}^{A^n}$, *and an operation* f *on* A *is a term operation of* $\mathfrak{A}$ *if and only if it preserves all subuniverses* $X_{T^{(n)}(\mathfrak{A})}$ $(n \geq 1)$.

PROOF. Let $C = \mathrm{Pol}_A\{X_{T^{(n)}(\mathfrak{A})} : n \geq 1\}$. By Lemma 1.2 we have $T(\mathfrak{A}) \subseteq C$, implying also that $X_{T^{(n)}(\mathfrak{A})}$ is a subuniverse of $\mathfrak{A}^{A^n}$ for every $n \geq 1$. To prove the reverse inclusion $\mathcal{T}(\mathfrak{A}) \supseteq C$, let $g \in C$, say g is n-ary. Then g preserves $X_{\mathcal{T}^{(n)}(\mathfrak{A})}$, so applying Lemma 1.2 for the n-ary projections $e_{n,1},\ldots,e_{n,n} \in \mathcal{T}^{(n)}(\mathfrak{A})$ we get that $g = g(e_{n,1},\ldots,e_{n,n}) \in \mathcal{T}^{(n)}(\mathfrak{A})$. This completes the proof.

COROLLARY 1.4. *For a finite algebra* $\mathfrak{A} = (A;F)$, *an operation* g *on* A *is a term operation of* $\mathfrak{A}$ *if and only if it preserves the subuniverses of finite powers of* $\mathfrak{A}$.

PROOF. Let S denote the set of subuniverses of finite powers of $\mathfrak{A}$. Since $\{X_{\mathcal{T}^{(n)}(\mathfrak{A})} : n \geq 1\} \subseteq S$, we have

$$\mathrm{Pol}_A\{X_{\mathcal{T}^{(n)}(\mathfrak{A})} : n \geq 1\} \supseteq \mathrm{Pol}_A\, S.$$

The left hand side equals $\mathcal{T}(\mathfrak{A})$ by Proposition 1.3. The right hand side contains F by definition, so it contains $\mathcal{T}(\mathfrak{A})$ as well by Proposition 1.1(a). Thus $\mathcal{T}(\mathfrak{A}) = \mathrm{Pol}_A\, S$, as claimed.

If $\mathfrak{A}$ is infinite, then $|A^n| = |A|$ for all $n \geq 1$, so the same argument as above yields

COROLLARY 1.5. *For an infinite algebra* $\mathfrak{A} = (A;F)$, *an operation* g *on* A *is a term operation of* $\mathfrak{A}$ *if and only if it preserves the subuniverses of* $\mathfrak{A}^A$.

Proposition 1.3 for A finite and Corollary 1.4 were proved independently by D. Geiger [1968] and V. G. Bodnarchuk, L. A. Kaluzhnin, V. N. Kotov, B. A. Romov [1969], and they were generalized to the infinite case by I. G. Rosenberg [1972].

For comparison, it is worth mentioning how those operations g on A can be characterized which preserve the subuniverses of finite powers of an infinite algebra $\mathfrak{A} = (A;F)$.

DEFINITION. Let C be a clone on A. We say that an operation $g \in O_A^{(n)}$ can be interpolated by operations from C iff for every finite subset B of A^n there exists an operation $f \in C$ such that $f|_B = g|_B$. The clone C is called *locally closed* iff it contains every operation that can be interpolated by operations from C. For an algebra $\mathfrak{A} = (A;F)$, the operations that can be interpolated by operations from $T(\mathfrak{A})$ [$P(\mathfrak{A})$] are called *local term operations* of $\mathfrak{A}$ [*local polynomial operations* of $\mathfrak{A}$, respectively].

The definitions immediately imply

PROPOSITION 1.6. *Let* A *be a set.*

(a) *The local term operations of any algebra* $\mathfrak{A} = (A;F)$ *form a locally closed clone on* A.

(b) $\mathrm{Pol}_A\, S$ *is a locally closed clone for arbitrary set* S *of subsets of finite powers of* A.

For arbitrary set T of n-ary operations on A, and for arbitrary finite subset B of A^n, we let $X_{T,B}$ denote the subset of A^B consisting of the functions $g|_B$ with $g \in T$, considered as elements of A^B. Similarly to Lemma 1.2 we have

LEMMA 1.7. *Let* T *be a set of* n*-ary operations on* A *and let* B *be a finite subset of* A^n. *A* k*-ary operation* g *on* A *preserves* $X_{T,B}$ *if and only if for all operations* $f_1,\dots,f_k \in T$, *there exists* $f \in T$ *such that* $g(f_1,\dots,f_k)|_B = f|_B$.

Thus we get the following analogue of Proposition 1.3 for local term operations.

PROPOSITION 1.8. *Let* $\mathfrak{A} = (A;F)$ *be an algebra. For every integer* $n \geq 1$ *and for arbitrary finite subset* B *of* A^n, $X_{T^{(n)}(\mathfrak{A}),B}$ *is a subuniverse of* $\mathfrak{A}^B$, *and an operation* f *on* A *is a local term operation of* $\mathfrak{A}$ *if and only* *if it preserves all these subuniverses* $X_{T^{(n)}(\mathfrak{A}),B}$.

PROOF. Let $C = \mathrm{Pol}_A\{X_{T^{(n)}(\mathfrak{A}),B} : n \geq 1, B \subseteq A^n, B$ is finite$\}$. In the same way as in Proposition 1.3, making use of Lemma 1.7 in place of Lemma 1.2, we can conclude that C is the clone of local term operations of $\mathfrak{A}$. The details are left to the reader.

As before, we get the required characterization, which is due to B. A. Romov [1977].

COROLLARY 1.9. *For an algebra* $\mathfrak{A} = (A;F)$, *an operation* g *on* A *is a local term operation of* $\mathfrak{A}$ *if and only if it preserves the subuniverses of finite powers of* $\mathfrak{A}$.

Clearly, if $\mathfrak{A}$ is finite, then the local term operations of $\mathfrak{A}$ are term operations as well, so Corollary 1.4 is a special case of Corollary 1.9.

For a set F of operations on a fixed set A, let $\mathrm{Inv}_A F$ denote the family of subuniverses of finite powers of the algebra $(A;F)$.

EXERCISE 1.10. For every set A, the mappings

$$F \mapsto \mathrm{Inv}\, F$$

$$\mathrm{Pol}\, S \leftarrow S$$

between the power sets of $\mathcal{O}_A$ and $\{B: B \subseteq A^n$ for some $n \geq 1\}$ define a Galois connection (or polarity), that is, the following two conditions hold for arbitrary subsets F, F' of $\mathcal{O}_A$ and for arbitrary sets S, S' of subsets of finite powers of A:

(1) if $F \subseteq F'$ then $\mathrm{Inv}\, F \supseteq \mathrm{Inv}\, F'$, and similarly, if $S \subseteq S'$ then $\mathrm{Pol}\, S \supseteq \mathrm{Pol}\, S'$;

(2) $F \subseteq \mathrm{Pol}\,\mathrm{Inv}\, F$ and $S \subseteq \mathrm{Inv}\,\mathrm{Pol}\, S$.

This implies also that for F, S as above we have

(3) $\operatorname{Inv} F = \operatorname{Inv} \operatorname{Pol} \operatorname{Inv} F$ and $\operatorname{Pol} S = \operatorname{Pol} \operatorname{Inv} \operatorname{Pol} S$.

Consequently

(4) the mappings Pol Inv and Inv Pol are closure operators, hence the respective closed sets (that is the sets of the form Pol S and those of the form Inv F, respectively) constitute complete lattices; moreover, Pol and Inv, when restricted to these lattices, yield mutually inverse dual isomorphisms.

Propositions 1.6 and 1.8 (or Corollary 1.9) show that the sets of the form $\operatorname{Pol}_A S$ are exactly the locally closed clones on A, or for A finite, they are exactly the clones on A. We note that the sets of the form $\operatorname{Inv}_A F$ can also be described as sets of subsets of finite powers of A that are closed under certain constructions. (For A finite this was discovered independently by V. G. Bodnarchuk, et al [1969] and D. Geiger [1968]. Later these results were generalized to the infinite case independently by R. Pöschel [1979] and L. Szabó [1978].) However, as we will not need this fact later on, we do not go into the details. Nevertheless, some constructions will be used quite often, so we introduce the notations here.

Let B be a subset of A^k $(k \geq 1)$. We will write $\underline{\underline{k}}$ for the set $\{1,\ldots,k\}$ indexing the components of B. For an ℓ-tuple $(i_1,\ldots,i_\ell) \in \underline{\underline{k}}^\ell$ we define the projection of B onto its components $i_1,\ldots,i_\ell$ by

$$\operatorname{pr}_{i_1,\ldots,i_\ell} B = \{(x_{i_1},\ldots,x_{i_\ell}) : (x_1,\ldots,x_k) \in B\}.$$

In particular, if $\ell = k$ and $i_1,\ldots,i_k$ is a permutation of $1,\ldots,k$, then $\operatorname{pr}_{i_1,\ldots,i_k} B$ arises from B by rearranging the components. The property that, up to the order of their components, the subsets B and B' of A^k coincide, will be denoted by $B \approx B'$. For example, if $k = 2$, then $\operatorname{pr}_{2,1} B$ is the inverse of B, which will be denoted by $B^{\smile}$. For a nonvoid subset I of $\underline{\underline{k}}$ with $I = \{i_1,\ldots,i_\ell\}$, $i_1 < \ldots < i_\ell$, we write $\operatorname{pr}_I B$ for $\operatorname{pr}_{i_1,\ldots,i_\ell} B$. The symbol

$B \preccurlyeq B_1 \times \ldots \times B_k$ will be used to designate that $pr_i\, B = B_i$ for all $1 \le i \le k$. The sign $\prec$ stands for "$\preccurlyeq$ and $\ne$". For $B \preccurlyeq B_1 \times \ldots \times B_k$ and for arbitrary bijections $\pi_i\colon B_i \to C_i$ $(C_i \subseteq A,\ 1 \le i \le k)$ we set

$$B[\pi_1,\ldots,\pi_k] = \{(x_1\pi_1,\ldots,x_k\pi_k)\colon (x_1,\ldots,x_k) \in B\}.$$

If $1 \le \ell < k$ and $a = (a_{\ell+1},\ldots,a_k) \in A^{k-\ell}$, then we define the subset of A^ℓ arising from B by "substituting the constants $a_{\ell+1},\ldots,a_k$ for the $(\ell+1)$-st up to the k-th components" as follows:

$$\begin{aligned} B(x_1,\ldots,x_\ell,a) &= B(x_1,\ldots,x_\ell,a_{\ell+1},\ldots,a_k) \\ &= \{(x_1,\ldots,x_\ell) \in A^\ell\colon (x_1,\ldots,x_\ell,a_{\ell+1},\ldots,a_k) \in B\}. \end{aligned}$$

As usual, for $C,C' \subseteq A^2$ we set

$$C \circ C' = \{(x,y) \in A^2\colon (x,z) \in C \text{ and } (z,y) \in C' \text{ for some } z \in A\}.$$

Let now $\mathfrak{A} = (A;F)$ be an arbitrary algebra. It is easy to check that $C \circ C'$ is a subuniverse of $\mathfrak{A}^2$ if C and C' are such. Similarly, $pr_{i_1,\ldots,i_\ell}\, B$ is a subuniverse of $\mathfrak{A}^\ell$ whenever B is a subuniverse of $\mathfrak{A}^k$ and $(i_1,\ldots,i_\ell) \in \underline{k}^\ell$. Furthermore, if $B \preccurlyeq B_1 \times \ldots \times B_k$ is a subuniverse of $\mathfrak{A}^k$ and for each bijection $\pi_i\colon B_i \to C_i$ $(C_i \subseteq A,\ 1 \le i \le k)$, $\pi_i^\square$ is a subuniverse of $\mathfrak{A}^2$, then $B[\pi_1,\ldots,\pi_k]$ is also a subuniverse of $\mathfrak{A}^k$. In general, the remaining construction fails to have the analogous property. However, if $\mathfrak{A}$ is idempotent, then $B(x_1,\ldots,x_\ell,a_{\ell+1},\ldots,a_k)$ is a subuniverse of $\mathfrak{A}^\ell$ whenever B is a subuniverse of $\mathfrak{A}^k$ and $a_{\ell+1},\ldots,a_k$ are arbitrary elements from A.

For later use and to illustrate the power of the above tools we finally prove a result establishing a relation between the subuniverses of finite powers of a finite algebra and its full idempotent reduct. We will call a subset B of A^k *irredundant* iff $pr_{i,j}\, B \nsubseteq \Delta_A$ for all $1 \le i < j \le k$ and $|pr_i\, B| > 1$ for all $1 \le i \le k$.

PROPOSITION 1.11. *Let* $\mathfrak{A} = (A;F)$ *be a finite algebra and* $\mathfrak{A}_o$ *its full idempotent reduct. Every set of the form* $\mathrm{pr}_{\underline{n}}\, B(x_1,\dots,x_\ell,a)$, *where* B *is a subuniverse of* $\mathfrak{A}^k$, $1 \le n \le \ell \le k$ *and* $a \in A^{k-\ell}$, *is a subuniverse of* $\mathfrak{A}_o^n$. *Conversely, all irredundant subuniverses of finite powers of* $\mathfrak{A}_o$ *are of this form.*

PROOF. The first claim is easy to check. To prove the second one, consider an irredundant subuniverse S of $\mathfrak{A}_o^n$ $(n \ge 1)$, and let $|S| = m$, $S = \{(s_{i1},\dots,s_{in}) : 1 \le i \le m\}$. Since S is irredundant, the elements $(s_{1j},\dots,s_{mj}) \in A^m$ $(1 \le j \le n)$ are pairwise distinct, and none of them equals $(a,\dots,a) \in A^m$ for any $a \in A$. Thus, if in the subuniverse $X_{T^{(m)}(\mathfrak{A})}$ of $\mathfrak{A}^{A^m}$ (cf. Proposition 1.3) we first substitute a in the component corresponding to $(a,\dots,a) \in A^m$ for every $a \in A$, and then project the resulting set onto its components $(s_{1j},\dots,s_{mj}) \in A^m$ $(1 \le j \le n)$, then we get the set

$$S' = \{(g(s_{11},\dots,s_{m1}),\dots,g(s_{1n},\dots,s_{mn})) : g \in T^{(m)}(\mathfrak{A}_o)\}.$$

Because of the projections we have $S \subseteq S'$. On the other hand, $S' \subseteq S$, since S is a subuniverse of $\mathfrak{A}_o^n$. Thus $S = S'$, concluding the proof.

<u>Minimal clones and maximal clones</u>

It is easy to see that for every set A, the lattice $\mathrm{Lat}(A)$ of subclones of $\mathcal{O}_A$ is an algebraic lattice. The atoms [dual atoms] of $\mathrm{Lat}(A)$ are called *minimal* [*maximal*] *clones*; that is, a subclone C of $\mathcal{O}_A$ is minimal iff $C \ne \mathcal{J}_A$ and C, $\mathcal{J}_A$ are the only subclones of C, while C is maximal iff $C \ne \mathcal{O}_A$ and C, $\mathcal{O}_A$ are the only clones containing C. We prove that for a finite set A, the lattice $\mathrm{Lat}(A)$ is atomic and dually atomic with finitely many atoms and dual atoms.