Lecture Notes in Mathematics

Edited by A. Dold, Heidelberg and B. Eckmann, Zürich

396

Karl Heinrich Hofmann
Michael Mislove
Albert Stralka

The Pontryagin Duality of Compact O-Dimensional Semilattices and its Applications

Springer-Verlag
Berlin · Heidelberg · New York 1974

Karl Heinrich Hofmann
Michael Mislove
Dept. of Mathematics
Tulane University
New Orleans, LA 70018/USA

Albert Stralka
Dept. of Mathematics
University of California
Riverside, CA 94202/USA

AMS Subject Classifications (1970): Primary: 06-02, 06A20, 20-02, 22A15
Secondary: 20M10

ISBN 3-540-06807-4 Springer-Verlag Berlin · Heidelberg · New York
ISBN 0-387-06807-4 Springer-Verlag New York · Heidelberg · Berlin

Offsetdruck: Julius Beltz, Hemsbach/Bergstr.

TO ALFRED HOBLITZELLE CLIFFORD

on his 65th birthday on the

11th of July, 1973

ACKNOWLEDGEMENTS

The authors were supported by NSF-Grants GP - 28655-A-1 and GP - 33912 during the preparation of this work. They are particularly grateful for the fact that this allowed Stralka to visit Tulane for a short period during which the general project of this research was outlined in a series of seminar sessions. The principal results were presented at the Houston conference on Lattice Theory in March, 1973. We also thank Mrs. Meredith R. Mickel for the excellent preparation of the typescript.

TABLE OF CONTENTS

INTRODUCTION

When Pontryagin established the duality between discrete and compact abelian groups in 1932 he was motivated by rather specific applications, mostly arising in an attempt at a general theory relating the following two examples from algebraic topology. Cech's homology groups of a compact space appeared as inverse limits of homology groups of finite complexes and thus behaved like compact abelian groups, whereas the discrete Cech cohomology groups arose from direct limits. The duality theory, however, evolved rather quickly to a rich structure theory which was applied to numerous areas of algebra, topology and analysis. In algebra and number theory these applications reach from Pontryagin's classification of the locally compact connected fields to the modern presentation of algebraic number theory (see W-1) while in group theory itself a rich interplay between the theory of abelian groups and compact groups developed giving impulses to both lines of research. Harmonic analysis, which had seen a great deal of activity during the twenties, was provided with precisely the right abstract tools by Pontryagin duality theory, and harmonic analysis became inseparable from the duality of locally compact abelian groups. Other dualities for various classes of topological groups followed, exemplified by the work of Tannaka and Krein in the thirties and forties, and the process of finding duality theories for general locally compact groups is still not completed.

In the theory of topological semigroups, which is less classical, duality theories have only lately been systematically investigated [H-4]. This is partly due to the fact that duality theories in the context of various classes of compact topological semigroups say, either do not exist, or, where indeed they do exist, are technically involved and rarely as simply expressed as Pontryagin duality [H-4]. However, in the case of semilattices, it was observed relatively early by Austin, as least on the object level, that an analog of Pontryagin character theory works [A-2], and this has been discussed in increasing measure by other authors (Baker and Rothman [B-1], Bowman [B-10], Hofmann [H-4], Schneperman [S-4]). In the meantime, duality theories for lattices and topological spaces in terms of characters were discussed by numerous authors in various degrees of generality; for a systematic treatment and further references in this direction see Hofmann and Keimel [H-5].

Nevertheless, it is pretty apparent, that the duality theory between discrete and compact semilattices, although having been treated from various angles, has never been systematically exploited. No applications to and from lattice theory have been made, and the theory of compact semigroups has not been brought to bear on this duality. The duality of semilattices should be a point where different lines of investigation merge: the algebraic theory of semilattices and lattices on one hand, and the theory of compact topological semigroups on the other.

In the following we make an attempt to present the duality theory of compact semilattices in this spirit. There are many features of this duality which place it in close parallel with Pontryagin duality; there are others in which it exhibits drawbacks, but also advantages.

We develop the general theory of the duality and we present applications to lattice theory on one hand and to compact monoid theory on the other. We hope that the future will bring applications beyond those which we know of and are able to discuss.

Various portions of the contents will be familiar to some groups of readers, varying along with the content; yet even what is likely to appear familiar is probably presented in a new light and, sometimes, in a more systematic fashion than any treatment other than that by duality would allow. What is perhaps well known to the person working in lattice theory may provide some new aspect for the worker in compact topological monoids, and vice versa. But even on familiar ground some new results emerge here and there.

The language of category theory provides a convenient and elegant medium for duality theory, and it places the emphasis where it belongs: on objects and functions equally. In the more conventional treatments of representations of semigroups or lattices, the role of the morphism is all to often ignored. However, the material entering from category theory is on the level of the theory of limits (touching upon the idea of Kan extensions of functors, although we will not explicitly speak about them) and the theory of adjoints. Very little of the deeper aspects of the theory of compact monoids will be needed, although the spirit of compact topological algebra pervades the discussion. Most of the lattice theory which appears is treated in a self-contained fashion, much of it with unconventional proofs.

The material is presented in the following fashion: In a preliminary Chapter, called Preliminaries, we provide the functorial language in which we will prove the duality. This involves functor categories, limit and colimit functors, and the concepts of density and codensity in categories. Beginning seriously with our topic, we open Chapter I with a section on the category of semilattices (with identity) and its basic properties; we then parallel this discussion in Section 2 by an analogous treatment of the category of compact zero dimensional semilattices (with identity). In Section 3 we prove the duality theorem; with our preparations the proof is very short and very apropos: We need to know that in both categories the finite objects determine the category in a sense which we make precise in terms of functorial density, and that the duality holds for finite semilattices. We do not need any intrinsic structural information about discrete or compact zero dimensional semilattices. This proof parallels a proof of the Pontryagin duality theorem for locally compact abelian groups which was recently given by Roeder [R-3]. Just as in the case of groups there are proofs of the duality theorem for semilattices available which utilize a considerable amount of structural information on semilattices. We prefer to present the proof which reflects the general set-up of numerous duality theorems, and then to go into the structural details for their own merit at a later time. In fact, this happens in Chapter II. In the first section we apply general principles from the theory of compact monoids to compact zero dimensional semilattices. In particular we record the monotone convergence theorem and the resulting long standing observation, that the underlying semilattice of a compact topological semilattice is a complete lattice. We introduce the concepts of a local minimum, a strong local minimum, and a semiminimum in a quasi-ordered set with a topology and their dual concepts. A local minimum is, as the name suggests, a minimum in one of its neighborhoods, a strong local minimum is an element which generates an open filter (upper set) and a semiminimum is a minimal element of the complement of some strong local maximum. In a topological semilattice, every local minimum is strong, but not every local maximum is strong. One of the important facts observed in this section is that the set $K(S)$ of local minima of a topological compact zero dimensional semilattice S is dense and closed under finite sups; in fact $s = \sup K(S) \cap Ss$ for each $s \in S$. It is also correct that every element is the inf of all semimaxima which dominate it. Thus there is an abundance of local minima and semimaxima. While there are plenty of local maxima, they do not play a role, and it may very well be the case that 1 is the only strong local maximum. On the other hand, there are also semilattices in which the set of local minima is dense but which nevertheless are not zero dimensional. In the second section of Chapter II we relate the concepts of characters and filters of a semilattice. In fact, beginning with a discrete semilattice, we set up an isomorphism between the compact zero dimensional character semilattice and the ∩-semilattice of all filters with a suitable topology. In this specific sense the character semilattice of a discrete semilattice is precisely the filter completion. As in all of our discussions we emphasize the functorial nature of such constructions. The third section of Chapter II is in many respects the core of the entire presentation. In the preceding section we described an alternative view of the characters of a discrete semilattice. Here we give an alternative description of a character of a compact zero dimensional discrete semilattice S; indeed there is an isomorphism between the character semilattice of S and the sup-semilattice K(S). The usefulness of this result is enhanced by a characterization theorem for the elements of K(S). Indeed an element k is a local minimum iff it is a compact element of the underlying complete lattice (whereby an element k of a lattice is compact if any relation $k \leq \sup X$ for a subset X of the lattice implies the existence of a finite subset $F \subseteq X$ with $k \leq \sup F$). Moreover this is equivalent to the statement that k is isolated in its principal ideal Sk and also to the statement that, in effect, it cannot be reached in Sk by any chain in $Sk \setminus \{k\}$. Recalling that a lattice is algebraic iff it is complete and every element is the sup of the compact elements which it dominates, we have now observed that the underlying lattice of any compact zero dimensional semilattice is an algebraic lattice. But we also demonstrate that, conversely, on every algebraic lattice there is a unique compact topology making it into a topological semilattice, and in this topology it is zero dimensional and the compact elements are precisely the local minima. Since we insist on the functorial aspect we have to find an algebraic description of the continuous morphisms: Indeed a semilattice morphism between two algebraic lattices is continuous relative to the unique topology just mentioned iff it is a lattice morphism preserving arbitrary infs and sups of chains. We call such morphisms algebraically continuous. Thus the principal results of this section and the Chapter may be summarized as follows: The category of compact zero dimensional semilattices (with identities) and continuous (identity preserving) semilattice morphisms is isomorphic to the category of algebraic lattices and algebraically continuous lattice maps and is dual to the category of all semilattices (with identities) and (identity preserving) semilattice morphisms. This result provides an algebraic basis for the entire theory. The question of a characterization of zero dimensional compact topological lattices (i.e. such objects in which the sup operation is also continuous) receives a partial answer in this section, too. This condition is satisfied if there is an abundance of strong maxima; unfortunately this condition is not purely algebraic, since there is no precise correspondence between the strong local maxima and the cocompact elements, and no equivalent formulation in terms of duality is known to us. However, as we will show in the later parts of the development, the situation becomes completely satisfactory in the presence of distributivity.

Chapters I and II contain the duality theory both in its general, i.e. category theoretical, and in its structural aspects. The remainder is devoted to applications. Chapter III links the theory with classical segments of lattice theory. The first Section of this Chapter is concerned with prime elements and distributivity; to no one's great surprise, these two concepts appear together. The concept of prime elements does not present any difficulty whatsoever in a semilattice, the concept of distributivity does. A semilattice has been called distributive if $\sup\{a,b\}x = \sup\{ax,bx\}$ whenever $\sup\{a,b\}$ exists. The nice aspect of this concept of distributivity is that it is characterized by the embeddability of the semilattice in a distributive lattice under preservation of existing sups. For the purpose of duality, however, this concept of distributivity is too weak, and we therefore call it the weak distributivity of a semilattice. Since distributivity involves sups we wish to bring the "virtual sups" into the play which exist in every semilattice (with identity), namely the filter $\uparrow a \cap \uparrow b$ where $\uparrow X$ denotes the filter of all elements dominating some element $x \in X$. We therefore define a semilattice to be distributive iff $\uparrow(\uparrow a \cap \uparrow b)x = \uparrow ax \cap \uparrow bx$ for all a,b,x in the semilattice. Every distributive semilattice is weakly distributive, the converse is false. One of the principal theorems of this section contains the result that a semilattice is distributive if and only if its (compact) character semilattice is distributive (and hence has a Brouwerian algebraic lattice as underlying semilattice). This is also equivalent to the property in the character semilattice that every element is the inf of the set of primes dominating it. On the other hand, a semilattice is primally generated in the sense that every element is a finite product (inf) of primes if and only if its character semilattice is a compact topological distributive lattice (i.e. has continuous sup-operation). This property of the character semilattice will be characterized in numerous other ways, the most purely algebraic being that its underlying lattice is bialgebraic, where we call a lattice bialgebraic if it is algebraic and if the opposite lattice (obtained by reversing the order) is also algebraic. The proofs of these facts are obtained through character theory. We define the concept of a sup-character (a special case of a sup-morphism between semilattices) which is a particular type of morphism preserving "virtual sups" (in the same spirit as we have used "virtual sups" to define distributivity). It turns out that a character of a semilattice is a sup-character iff it is a prime element of the character semilattice. This is the main link via duality between primes and distributivity, since distributivity implies the separation of points by sup-characters. However, one should be warned against assuming the converse; we are unable to prove it or furnish a counter example. Some of the principal results of the section may be summarized as follows: The category of distributive semilattices and morphisms mapping primes into primes is dual to the category of Brouwerian algebraic lattices and lattice morphisms preserving arbitrary sups and infs. The category of primally generated semilattices and prime preserving morphisms is dual to the category of Brouwerian bialgebraic lattices and lattice morphisms preserving arbitrary sups and infs (which category is isomorphic to the category of compact zero dimensional distributive topological lattices and continuous lattice morphisms). Some supplementary results link these facts with the categories of partially ordered sets and certain categories of topological spaces (the so-called spectral spaces). Section 2 sheds some light on the relation between the duality theory of semilattices and the classical theory of Boolean lattices. We prove that being Boolean and being free are dual properties in the following sense: The character semilattice of a free semilattice is a compact zero dimensional Boolean lattice with continuous multiplication (inf-operation); this implies in particular that every such object is of the form 2^X for some set X in the product topology. If, on the other hand, we start with a Boolean lattice and find the character semilattice of the underlying inf semilattice, then it turns out to be a free compact zero dimensional semilattice over a compact zero dimensional space. We say that a morphism between Boolean objects is Boolean iff it preserves complements. Since its dual is a morphism between free objects (in the sense explained) one naturally asks how these dual morphisms are characterized. In the case of free discrete semilattices, they are precisely the set induced morphisms; in the case of the free compact zero dimensional semilattices, they are precisely the space induced morphisms.

The third section of the chapter describes the projectives and injectives in the category of semilattices and its dual. These results complement the characterization theorems of Horn and Kimura, and the availability of duality enables us to give the proofs a different setting.

While Chapter III is focused on applications to algebra, the final Chapter IV illustrates various applications to topology, specifically the theory of compact topological semilattices and monoids. Since, in a sense, compact monoids were our point of departure in Chapter II when we began the structural investigation, this closes the circle. In Section 1 we use duality to determine "the topological size" of a compact zero dimensional semilattice. There are, among numerous others, two cardinals which are particularly useful to determine the size of a topology: one is the so-called weight, i.e. the smallest cardinal of a basis for the topology, and the other is what we call the separability number, i.e. the smallest cardinal of a dense subset. For a compact zero dimensional semilattice S we show that the weight $w(S)$ is the cardinal of its character semilattice $\hat{S}$, which by earlier results, equals card $K(S)$. If $d(S)$ is the separability number, we prove $\log w(S) \leq d(S) \leq w(S)$ (where $\log a = \min\{b \mid a \leq 2^b\}$); examples show that strict inequality occurs in both cases, but the estimates are the best possible. This contrasts the situation for compact abelian groups G: There one has $w(G) = \mathrm{card}(\hat{G})$ just as in the case of semilattices, but the equality $d(G) = \log w(G)$ is always true. Section 2 is a report on the application of duality to a very important line of research in the theory of compact monoids: the investigation of quotient morphisms which raise topological dimension. In fact, with the aid of duality, we are able to give a complete characterization of those compact zero dimensional semilattices which have quotients of positive topological dimension. They are precisely those which have the Cantor set chain (under min) as a quotient; dually, they are precisely those whose character semilattice contains an order dense non-degenerate countable chain. The results are in fact much sharper. The proofs of these results appear elsewhere, and we content ourselves here with a descriptive discussion of this theory. In the third and final section we prove another parallel theorem to one established for groups by various authors, in the most general form and with the most direct proof by Archangelski [A-1]. The theorem for groups says that a topological group whose underlying space is extremally disconnected is necessarily discrete. Here we show (by entirely different methods) that an extremally disconnected compact semilattice is necessarily finite. As a corollary of both results we then obtain the following theorem on compact semigroups: A compact monoid S which is a union of groups and is such that the set of idempotents is commutative cannot be extremally disconnected unless it is finite. As a biproduct of the results of this section we have an immediate proof of the fact that the space of closed subsets of an infinite compact extremally disconnected space is never extremally disconnected, since such a space is always a compact topological semilattice under $\cup$.

CHAPTER 0. Preliminaries

Section 1. About dense subcategories and the extension of natural transformations.

The background material which we are going to prepare here for later use may be presented in various degrees of generality. We choose that level which leads to our applications in the most direct fashion and does not require an apparatus of an exorbitant grade of abstraction. For a vastly more general approach to some of the ideas used here see [I-2].

We have to consider the category of diagrams in a given category. Let us denote with $\underline{cat}$ the category whose objects are small categories and whose morphisms are functors between them. The following lemmas have straightforward verifications (which can become a bit technical).

LEMMA 1.1. Let $\underline{A}$ be a category. The following definitions yield a category $\underline{A}^{\underline{cat}}$:

a) Objects: Objects are functors $D : \underline{X} \to \underline{A}$, $\underline{X} \in ob(\underline{cat})$

b) Morphisms: Morphisms $D \to E$, $D : \underline{X} \to \underline{A}$, $E : \underline{Y} \to \underline{A}$ are given by pairs $(f,a) : D \to E$ such that $f : \underline{X} \to \underline{Y}$ is a functor (i.e. a morphism in $\underline{cat}$) and $a : D \to Ef$ is a natural transformation of functors $\underline{X} \to \underline{A}$.

c) Composition: $(f,a)(g,b) = (fg,(ag)b)$

LEMMA 1.2. If $\phi : \underline{A} \to \underline{B}$ is a functor into a complete category, then there is an induced functor

$$\phi' = \phi^{\underline{cat}} : \underline{A}^{\underline{cat}} \to \underline{B}^{\underline{cat}}$$

$$\phi'(F) = \phi \circ F, \ \phi'(a,f) = (\phi a,f).$$

LEMMA 1.3. If $\underline{A}$ is cocomplete, then there is a functor $COLIM : \underline{A}^{\underline{cat}} \to \underline{A}$ given by $COLIM(D) = colim\ D$ and $COLIM(a,f) : colim\ D \to colim\ E$ given by

$$\begin{array}{ccccc}
((colim\ D)f)_{\underline{Y}} & \longleftarrow & (colim\ Df)_{\underline{Y}} & \xleftarrow{(colim\ a)_{\underline{Y}}} & (colim\ E)_{\underline{Y}} \\
\uparrow \lambda^{D} & & \uparrow \lambda^{Df} & & \uparrow \lambda^{E} \\
Df & = & Df & \xleftarrow[a]{} & E
\end{array}$$

where for any object $A \in \underline{A}$, the constant functor $\underline{X} \to \underline{A}$ with value A is denoted by $A_{\underline{X}}$ and where $\lambda^{D} : D \to (colim\ D)_{\underline{X}}$ denotes the limit natural transformation.

We now single out a subcategory of $\underline{A}^{\underline{cat}}$ which is of relevance in our context.

DEFINITION 1.4. Let $\underline{dir} \subseteq \underline{cat}$ be the full subcategory of all directed sets $\underline{X}$ made into a small category by writing $x \to y$ iff $x \leq y$. Now $\underline{A}^{\underline{dir}}$ is the full subcategory in $\underline{A}^{\underline{cat}}$ of all direct systems, i.e. functors $D : \underline{X} \to \underline{A}$ with $\underline{X} \in ob(\underline{dir})$. □

In topology (sheaf theory) and topological algebra, in fact, wherever limits more than colimits play a role, a variant of this set-up is more important; in duality theory we need both versions.

LEMMA 1.5. Let $(\underline{A}^{\underline{cat}})^* = [(\underline{A}^{op})^{\underline{cat}}]^{op}$. The objects of this category are functors $D : \underline{X} \to \underline{A}$ again, its morphisms are pairs $(f,a) : E \to D$, $E : \underline{Y} \to \underline{A}$, $D : \underline{X} \to \underline{A}$ with a functor $f : \underline{X} \to \underline{Y}$ and a natural transformation $a : Ef \to D$ of functors $\underline{X} \to \underline{A}$, which are composed according to $(g,b)(f,a) = (fg, b(ag))$.

If $\underline{A}$ is complete, then there is a functor $LIM : (\underline{A}^{\underline{cat}})^* \to \underline{A}$ with $LIM\ (D) = lim\ D$ and with $LIM\ (f,a)$ defined in a fashion analogous to $COLIM\ (f,a)$ in 1.3.

This category contains a subcategory which will be of importance to us in a similar fashion as will be $\underline{A}^{\underline{dir}}$ in 1.4:

DEFINITION 1.6. Let $\underline{inv} \subseteq \underline{cat}$ be the full subcategory of all $\underline{X}^{op}$, $\underline{X} \in \underline{inv}$, i.e. the full subcategory of all inverse systems. Then $\underline{A}^{\underline{inv}}$ is the full subcategory in $(\underline{A}^{\underline{cat}})^*$.

Now we discuss briefly the concept of (directed) codensity and density.

DEFINITION 1.7. Let $\underline{A}_o$ be a subcategory of $\underline{A}$ with inclusion functor J. We say that $\underline{A}_o$ is codense in $\underline{A}$ (through direct limits) iff

i) $\underline{A}$ has direct limits (i.e. each $D \in \text{ob } \underline{A}^{\underline{dir}}$ has a colimit).

ii) there is a functor $\Delta : \underline{A} \to \underline{A}_o^{\underline{dir}}$ such that the functor

$$\underline{A} \xrightarrow{\Delta} \underline{A}_o^{\underline{dir}} \xrightarrow{J^{\underline{dir}}} \underline{A}^{\underline{dir}} \xrightarrow{\text{COLIM}} \underline{A}$$

is naturally isomorphic to the identity functor.

On the other hand, we say that $\underline{A}_o$ is dense in $\underline{A}$ (through projective limits) iff

i) $\underline{A}$ has projective limits (i.e. each $D \in \text{ob } \underline{A}^{\underline{inv}}$ has a limit).

ii) there is a functor $\nabla : \underline{A} \to \underline{A}_o^{\underline{inv}}$ such that the functor

$$\underline{A} \xrightarrow{\nabla} \underline{A}_o^{\underline{inv}} \xrightarrow{J^{\underline{inv}}} \underline{A}^{\underline{inv}} \xrightarrow{\text{LIM}} \underline{A}$$

is naturally isomorphic to the identity functor.□

We will encounter both of these situations in the context of our duality theory. The following is the relevant result we are using:

PROPOSITION 1.8. Let $\underline{A}_o$ be a codense [dense] subcategory of $\underline{A}$ (with inclusion functor J) and let $F,G : \underline{A} \to \underline{B}$ be two cocontinuous [continuous] (i.e. direct limit [projective limit] preserving) functors into a category with direct (projective) limits. Then for every natural transformation $\alpha_o : FJ \to GJ$ there is a unique natural transformation $\alpha : F \to G$ with $\alpha J = \alpha_o$ (i.e. α_o has a unique extension α).

Proof. Observe $F \cong F \text{ COLIM } J^{dir}\, \Delta \cong \text{COLIM}(FJ)^{dir}\, \Delta$ and define an isomorphism $\text{COLIM } \alpha_o{}^{dir}\, \Delta : \text{COLIM}(FJ)^{dir}\, \Delta \to \text{COLIM } (GJ)^{dir}\, \Delta$. For details see [H-3].

COROLLARY 1.9. Under the hypotheses of 1.8 if $\beta : F \to G$ is a natural transformation such that βJ is a natural isomorphism, then β itself is an isomorphism. In simplified terms: if the restrictions of two cocontinuous [continuous] functors to a codense [dense] subcategory are isomorphic, then they are isomorphic.

Let us note in passing that the theory outlined is a special instance of the theory of right and left Kan extensions of functors. This theory generally also provides the existence of extensions of functors, but we will not need this aspect here.